Non-linear Supersymmetry for non-Hermitian, non-diagonalizable Hamiltonians: II. Rigorous results

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Abstract

We continue our investigation of the non-linear SUSY for complex potentials started in the Part I [7] and prove the theorems characterizing its structure in the case of non-diagonalizable Hamiltonians. This part provides the mathematical basis of previous studies. The classes of potentials invariant under SUSY transformations for non-diagonalizable Hamiltonians are specified and the asymptotics of formal eigenfunctions and associated functions are derived. Several results on the normalizability of associated functions at infinities are rigorously proved. Finally the Index Theorem on relation between Jordan structures of intertwined Hamiltonians depending of the behavior of elements of canonical basis of supercharge kernel at infinity is proven.

1. Introduction: definitions and notation

In this part of the paper we continue the investigation of the nonlinear SUSY [1]–[6] (see, the extended list of references in [7]) for complex potentials started in the Part I [7] and prove the theorems characterizing its structure in the case of non-diagonalizable Hamiltonians. We use the class of potentials invariant under SUSY transformations for non-diagonalizable Hamiltonians and prove several results concerning the normalizability of associated functions at $+\infty$ or $-\infty$. These results allow to unravel the relation between Jordan cells in SUSY partner Hamiltonians which was described in the Index Theorem in Sec 6. of [7].

All the proofs and results of this part are safely applicable to PT symmetric non-Hermitian Hamiltonians [8, 9, 10] with soft type [7] of non-Hermiticity when the real part of a potential dominates over its imaginary one at both coordinate infinities. The latter property is embedded into the chosen classes of potentials.

Let us summarize this part of the paper aimed to derive Theorem 3 and Lemmas 1–4 discussed in Part I [7]. First we introduce the relevant classes of potentials K (main) and $\mathcal{K}$ (auxiliary) as well as we remind the notion of formal associated functions of a Hamiltonian. Next we provide necessary estimates for potentials belonging to the class $\mathcal{K}$ (Lemma 5) and for auxiliary integrals (Lemmas 6, 7). Furthermore, we derive the asymptotics of formal eigenfunctions (Lemma 8) and associated functions (Lemma 9) of Hamiltonians with potentials belonging to the class $\mathcal{K}$. Then the invariance of the classes $\mathcal{K}$ and K under intertwining is proved (Lemmas 10 and 1 respectively). And finally the proofs of the Lemma 2 (on properties of a sequence of formal associated functions under intertwining), of the Lemma 3 (on normalizability of elements of the canonical basis of

an intertwining operator), of the Lemma 4 (on interrelation in (non)normalizability of canonical bases of mutually transposed intertwining operators) and of the Theorem 3 (on relation between Jordan structures of intertwined Hamiltonians depending on the asymptotic behavior of elements of the canonical basis of an intertwining operator kernel at $\pm\infty$) will be presented. The enumeration of definitions and corollaries in brackets corresponds to the enumeration of the same definitions and corollaries in Part I [7].

In the paper we use the following classes of potentials.

Definition 1 (2). Let K be the set of all potentials $V(x)$ such that:

1) $V(x) \in C^{\infty}_{\mathbb{R}}$;

2) there are $R_0 > 0$ and $\varepsilon > 0$ (R_0 and ε depend on $V(x)$) such that for any $|x| \geqslant R_0$ the inequality $\operatorname{Re} V(x) \geqslant \varepsilon$ takes place;

3)
$$\operatorname{Im} V(x)/\operatorname{Re} V(x) = o(1), \qquad x \to \pm\infty; \tag{1}$$

4) functions
$$\Bigl(\int\limits_{\pm R_0}^{x} \sqrt{|V(x_1)|}dx_1\Bigr)^2 \Bigl(\frac{|V'(x)|^2}{|V(x)|^3} + \frac{|V''(x)|}{|V(x)|^2}\Bigr) \tag{2}$$

are bounded respectively for $x \geqslant R_0$ and $x \leqslant -R_0$.

Definition 2. Let $\mathcal{K}$ be the set of all potentials $V(x)$ such that:

1) $V(x)$ is a complex-valued (in particular, real-valued) function, defined on the real axis with possible exception at some points;

2) there are $R_0 > 0$ and $\varepsilon > 0$ (R_0 and ε depend on $V(x)$) such that $V(x)\Big|_{[R_0,+\infty[} \in C^2_{[R_0,+\infty[}$, $V(x)\Big|_{]-\infty,-R_0]} \in C^2_{]-\infty,-R_0]}$ and for any $|x| \geqslant R_0$ the following inequality holds:
$$\operatorname{Re} V(x) \geqslant \varepsilon; \tag{3}$$

3)
$$\operatorname{Im} V(x)/\operatorname{Re} V(x) = o(1), \qquad x \to \pm\infty; \tag{4}$$

4) functions
$$\Bigl(\int\limits_{\pm R_0}^{x} \sqrt{|V(x_1)|}dx_1\Bigr)^2 \Bigl(\frac{|V'(x)|^2}{|V(x)|^3} + \frac{|V''(x)|}{|V(x)|^2}\Bigr)$$

are bounded respectively for $x \geqslant R_0$ and $x \leqslant -R_0$.

Let us clarify that K is a main class of potentials — the class of physical potentials, and $\mathcal{K}$ is an auxiliary, wider class of potentials — the class, containing potentials of intermediate Hamiltonians (corresponding to factorization of an intertwining operator in the product of intertwining operators of first order in derivative) in the case when potentials of the initial and final Hamiltonians belong to K.

In what follows we shall use the functions of the form $(V(x) - \lambda)^{\varkappa}$, where $V(x) \in \mathcal{K}$, $\varkappa > 0$, $\lambda \in \mathbb{C}$ and either $\lambda \leqslant 0$ or $\operatorname{Im}\lambda \neq 0$. Branches of these functions will be identically selected by the condition
$$|\arg(V(x) - \lambda)| < \pi. \tag{5}$$
In the case $\lambda \leqslant 0$ this condition can be fulfilled in view of (3), and in the case $\operatorname{Im}\lambda \neq 0$ because of (4) the condition (5) can be satisfied for any $|x| > R_2$, where $R_2 \geqslant R_0$ is such that for any $|x| \geqslant R_2$ the inequality
$$\frac{|\operatorname{Im} V(x)|}{|\operatorname{Re} V(x)|} \leqslant \frac{1}{2}\frac{|\operatorname{Im}\lambda|}{|\operatorname{Re}\lambda|}, \qquad |x| \geqslant R_2 \tag{6}$$

holds. For $-R_0 < x < R_0$ (if $\lambda \leqslant 0$) or for $-R_2 < x < R_2$ (if $\operatorname{Im}\lambda \neq 0$) the functions of the form $(V(x)-\lambda)^{\varkappa}$ will not be used.

The notation is adopted,

$$\alpha(x;\lambda) = \frac{5}{16}\frac{(V'(x))^2}{(V(x)-\lambda)^{5/2}} - \frac{1}{4}\frac{V''(x)}{(V(x)-\lambda)^{3/2}}, \qquad \alpha(x) = \alpha(x;0),$$

$$\hat{\alpha}(x;\lambda) = \frac{5}{16}\frac{|V'(x)|^2}{|V(x)-\lambda|^{5/2}} + \frac{1}{4}\frac{|V''(x)|}{|V(x)-\lambda|^{3/2}}, \qquad \hat{\alpha}(x) = \hat{\alpha}(x;0),$$

$$\xi_{\uparrow\downarrow}(x;\lambda) = \pm\int\limits_{\pm R_1}^{x} \sqrt{V(x_1)-\lambda}\,dx_1, \qquad R_1 = \begin{cases} R_0, & \lambda \leqslant 0,\\ R_2, & \operatorname{Im}\lambda \neq 0,\end{cases}$$

$$\xi_{\uparrow\downarrow}(x) = \pm\int\limits_{\pm R_0}^{x} \sqrt{|V(x_1)|}\,dx_1, \quad I_{1,\uparrow\downarrow}(x;\lambda) = \pm\int\limits_{x}^{\pm\infty} \hat{\alpha}(x_1;\lambda)\,dx_1, \quad I_{1,\uparrow\downarrow}(x) = I_{1,\uparrow\downarrow}(x;0),$$

$$I_{2,\uparrow\downarrow}(x;\lambda) = \pm\int\limits_{x}^{\pm\infty} \hat{\alpha}(x_1;\lambda)e^{-2\operatorname{Re}(\xi_{\uparrow\downarrow}(x_1;\lambda)-\xi_{\uparrow\downarrow}(x;\lambda))}\,dx_1,$$

$$I_{3,\uparrow\downarrow}(x;\lambda) = \pm\int\limits_{\pm R_1}^{x} \hat{\alpha}(x_1;\lambda)e^{-2\operatorname{Re}(\xi_{\uparrow\downarrow}(x;\lambda)-\xi_{\uparrow\downarrow}(x_1;\lambda))}\,dx_1,$$

$$C = \max_{|x|\geqslant R_0}\frac{|\operatorname{Im}V(x)|}{|\operatorname{Re}V(x)|}.$$

The notion of a formal associated function, used in this paper, is defined as follows.

Definition 3 (1). The function $\psi_{n,i}(x)$ is called a formal associated function of i-th order of the Hamiltonian h for a spectral value λ_n, if

$$(h-\lambda_n)^{i+1}\psi_{n,i} \equiv 0, \qquad (h-\lambda_n)^{i}\psi_{n,i} \not\equiv 0, \tag{7}$$

where the adjective 'formal' emphasizes that a related function is not necessarily normalizable.

In particular, the associated function of zero order $\psi_{n,0}$ is a formal eigenfunction of h (a solution of the homogeneous Schrödinger equation, not necessarily normalizable).

In this paper we employ the normalizability of functions and in particular the normalizability at $+\infty$ (at $-\infty$), which is defined as follows.

Definition 4 (3). A function $f(x)$ is called normalizable at $+\infty$ (at $-\infty$), if there is R_+ (R_-) such that

$$\int\limits_{R_+}^{+\infty} |f(x)|^2\,dx < +\infty \qquad \Bigg(\int\limits_{-\infty}^{R_-} |f(x)|^2\,dx < +\infty\Bigg). \tag{8}$$

Otherwise $f(x)$ is called non-normalizable at $+\infty$ (at $-\infty$).

2. Estimates on potentials and asymptotics of useful integrals

Lemma 5. *If $V(x) \in \mathcal{K}$, $\lambda \in \mathbb{C}$ and either $\lambda \leqslant 0$ or $\operatorname{Im}\lambda \neq 0$, then there are constants C_1, C_2, $C_3 > 0$ such that for any $|x| \geqslant R_1$ the inequalities are valid,*

$$C_1 \leqslant \frac{|V(x)|^2}{|V(x)-\lambda|^2} \leqslant C_2, \tag{9}$$

$$\operatorname{Re}\sqrt{V(x)-\lambda} \geqslant C_3\sqrt{|V(x)|}. \tag{10}$$

Proof.

Let us first consider the case $\lambda \leqslant 0$. Then the right side of (9) is obvious. The left side of (9) follows from the chain

$$\frac{|V-\lambda|^2}{|V|^2} \equiv \frac{(1-\lambda/\operatorname{Re}V)^2 + \operatorname{Im}^2 V/\operatorname{Re}^2 V}{1+\operatorname{Im}^2 V/\operatorname{Re}^2 V} \leqslant (1-\frac{\lambda}{\varepsilon})^2 + C^2.$$

The inequality (10) is derived from the chain

$$\frac{|V|}{(\operatorname{Re}\sqrt{V-\lambda})^2} \equiv \frac{2\sqrt{1+\operatorname{Im}^2 V/\operatorname{Re}^2 V}}{1-\lambda/\operatorname{Re}V + \sqrt{(1-\lambda/\operatorname{Re}V)^2 + \operatorname{Im}^2 V/\operatorname{Re}^2 V}} \leqslant \frac{2\sqrt{1+C^2}}{1+1}.$$

Let us now consider the case $\operatorname{Im}\lambda \neq 0$. In this case the left side of (9) is provided by the chain

$$\frac{|V-\lambda|^2}{|V|^2} \equiv \frac{(1-\operatorname{Re}\lambda/\operatorname{Re}V)^2 + (\operatorname{Im}V/\operatorname{Re}V - \operatorname{Im}\lambda/\operatorname{Re}V)^2}{1+\operatorname{Im}^2 V/\operatorname{Re}^2 V} \leqslant (1+\frac{|\operatorname{Re}\lambda|}{\varepsilon})^2 + (C+\frac{|\operatorname{Im}\lambda|}{\varepsilon})^2.$$

The right side of (9) in the subcase $\operatorname{Re}V \geqslant 2|\operatorname{Re}\lambda|$ follows from the sequence of inequalities

$$\frac{|V|^2}{|V-\lambda|^2} \equiv \frac{1+\operatorname{Im}^2 V/\operatorname{Re}^2 V}{(1-\operatorname{Re}\lambda/\operatorname{Re}V)^2 + (\operatorname{Im}V/\operatorname{Re}V - \operatorname{Im}\lambda/\operatorname{Re}V)^2}$$

$$\leqslant \frac{1+C^2}{(1-|\operatorname{Re}\lambda|/\operatorname{Re}V)^2} \leqslant 4(1+C^2),$$

and in the subcase $\operatorname{Re}V \leqslant 2|\operatorname{Re}\lambda|$ from the sequence[1]

$$\frac{|V|^2}{|V-\lambda|^2} \equiv \frac{1+\operatorname{Im}^2 V/\operatorname{Re}^2 V}{(1-\operatorname{Re}\lambda/\operatorname{Re}V)^2 + (\operatorname{Im}V/\operatorname{Re}V - \operatorname{Im}\lambda/\operatorname{Re}V)^2} \leqslant$$

$$\frac{1+C^2}{(1-|\operatorname{Re}\lambda|/\operatorname{Re}V)^2 + (|\operatorname{Im}\lambda|/\operatorname{Re}V - |\operatorname{Im}\lambda|/(2|\operatorname{Re}\lambda|))^2} \equiv$$

$$\frac{1+C^2}{\frac{|\lambda|^2}{\operatorname{Re}^2\lambda}\left(\frac{|\operatorname{Re}\lambda|}{\operatorname{Re}V} - \frac{1}{2} - \frac{\operatorname{Re}^2\lambda}{2|\lambda|^2}\right)^2 + \frac{\operatorname{Im}^2\lambda}{4|\lambda|^2}} \leqslant \frac{4|\lambda|^2(1+C^2)}{\operatorname{Im}^2\lambda}.$$

One can obtain (10) with the help of the inequality

$$\frac{|V|}{(\operatorname{Re}\sqrt{V-\lambda})^2} \equiv \frac{2\sqrt{1+\operatorname{Im}^2 V/\operatorname{Re}^2 V}}{1-\operatorname{Re}\lambda/\operatorname{Re}V + \sqrt{(1-\operatorname{Re}\lambda/\operatorname{Re}V)^2 + (\operatorname{Im}V/\operatorname{Re}V - \operatorname{Im}\lambda/\operatorname{Re}V)^2}} \leqslant$$

[1]For the derivation of this chain the inequality (6) is used.

$$\frac{2\sqrt{1+C^2}}{1-|\mathrm{Re}\,\lambda|/\mathrm{Re}\,V+\sqrt{(1-|\mathrm{Re}\,\lambda|/\mathrm{Re}\,V)^2+(|\mathrm{Im}\,V|/\mathrm{Re}\,V-|\mathrm{Im}\,\lambda|/\mathrm{Re}\,V)^2}}. \tag{11}$$

In the subcase $\mathrm{Re}\,V \geqslant 2|\mathrm{Re}\,\lambda|$ the right side of (11) is less than or equal to $2\sqrt{1+C^2}$, wherefrom (10) follows, and in the subcase $\mathrm{Re}\,V \leqslant 2|\mathrm{Re}\,\lambda|$ the right side (11) is less than or equal to

$$\frac{2\sqrt{1+C^2}}{1-|\mathrm{Re}\,\lambda|/\mathrm{Re}\,V+\sqrt{(1-|\mathrm{Re}\,\lambda|/\mathrm{Re}\,V)^2+(|\mathrm{Im}\,\lambda|/\mathrm{Re}\,V-|\mathrm{Im}\,\lambda|/(2\mathrm{Re}\,\lambda))^2}} \equiv$$

$$\frac{2\sqrt{1+C^2}}{1-|\mathrm{Re}\,\lambda|/\mathrm{Re}\,V+(|\lambda|/|\mathrm{Re}\,\lambda|)\sqrt{(|\mathrm{Re}\,\lambda|/\mathrm{Re}\,V-1/2-\mathrm{Re}^2\lambda/(2|\lambda|^2))^2+\mathrm{Re}^2\lambda\mathrm{Im}^2\lambda/(4|\lambda|^4)}}. \tag{12}$$

Insofar as the function

$$f(y)=1-y+\frac{|\lambda|}{|\mathrm{Re}\,\lambda|}\sqrt{(y-\frac{1}{2}-\frac{\mathrm{Re}^2\lambda}{2|\lambda|^2})^2+\frac{\mathrm{Re}^2\lambda\mathrm{Im}^2\lambda}{(4|\lambda|^4)}}$$

has a minimum at the point $y=1/2+\mathrm{Re}^2\lambda/|\lambda|^2$, (12) is less than or equal to

$$\frac{2\sqrt{1+C^2}}{1-\mathrm{Re}^2\,\lambda/|\lambda|^2+(|\lambda|/|\mathrm{Re}\,\lambda|)\sqrt{\mathrm{Re}^4\lambda/(4|\lambda|^4)+\mathrm{Re}^2\lambda\mathrm{Im}^2\lambda/(4|\lambda|^4)}} \equiv \frac{2|\lambda|^2}{\mathrm{Im}^2\lambda}\sqrt{1+C^2}.$$

Thus, Lemma 5 is proved.

Corollary 1. If $V(x)\in\mathcal{K}$, $\lambda\in\mathbb{C}$ and either $\lambda\leqslant 0$ or $\mathrm{Im}\,\lambda\neq 0$, then

$$\frac{|V'(x)|}{|V(x)-\lambda|^{3/2}}=O\Big(\frac{1}{\xi_{\uparrow\downarrow}(x)}\Big), \qquad x\to\pm\infty.$$

Lemma 6. *If $V(x)\in\mathcal{K}$, $\lambda\in\mathbb{C}$ and either $\lambda\leqslant 0$ or $\mathrm{Im}\,\lambda\neq 0$, then for any $|x|\geqslant R_1$ the integrals $I_{1,\uparrow\downarrow}(x;\lambda)$ and $I_{2,\uparrow\downarrow}(x;\lambda)$ converge and the estimates hold:*

$$I_{1,\uparrow\downarrow}(x;\lambda)=O\Big(\frac{1}{\xi_{\uparrow\downarrow}(x)}\Big), \qquad x\to\pm\infty,$$

$$I_{2,\uparrow\downarrow}(x;\lambda)=O\Big(\frac{1}{\xi^2_{\uparrow\downarrow}(x)}\Big), \qquad x\to\pm\infty,$$

$$I_{3,\uparrow\downarrow}(x;\lambda)=O\Big(\frac{1}{\xi^2_{\uparrow\downarrow}(x)}\Big), \qquad x\to\pm\infty.$$

Proof.

Because the proofs for the cases $x\to+\infty$ and $x\to-\infty$ are similar, we shall consider the case $x\to+\infty$ only. Due to $V\in\mathcal{K}$ and Lemma 5 there are positive constants C_4, …, C_9 and ξ_0 such that

$$I_{1,\uparrow}(x;\lambda)\leqslant C_4\int_x^{+\infty}\frac{|V'|^2}{|V|^{5/2}}dx_1+C_5\int_x^{+\infty}\frac{|V''|}{|V|^{3/2}}dx_1\leqslant C_6\int_x^{+\infty}\frac{\sqrt{|V|}}{\xi_\uparrow^2}dx_1\equiv C_6\int_x^{+\infty}\frac{\xi'_\uparrow}{\xi^2_\uparrow}dx_1=\frac{C_6}{\xi_\uparrow(x)},$$

$$I_{2,\uparrow}(x;\lambda)\leqslant C_7\int_x^{+\infty}\frac{\xi'_\uparrow(x_1)}{\xi^2_\uparrow(x_1)}e^{-2\int_x^{x_1}\mathrm{Re}\,\sqrt{V(x_2)-\lambda}dx_2}dx_1\leqslant C_7\int_x^{+\infty}\frac{\xi'_\uparrow(x_1)}{\xi^2_\uparrow(x_1)}e^{-C_8\int_x^{x_1}\sqrt{|V(x_2)|}dx_2}dx_1\equiv$$

$$C_7\int\limits_x^{+\infty}\frac{\xi'_\uparrow(x_1)}{\xi^2_\uparrow(x_1)}e^{C_8(\xi_\uparrow(x)-\xi_\uparrow(x_1))}dx_1\leqslant\frac{C_7}{\xi^2_\uparrow(x)}e^{C_8\xi_\uparrow(x)}\int\limits_x^{+\infty}\xi'_\uparrow(x_1)e^{-C_8\xi_\uparrow(x_1)}dx_1=\frac{C_7/C_8}{\xi^2_\uparrow(x)},$$

$$I_{3,\uparrow}(x;\lambda)\leqslant C_9\int\limits_{R_0}^{x}\frac{\xi'_\uparrow(x_1)}{(\xi_0+\xi_\uparrow(x_1))^2}e^{-C_8(\xi_\uparrow(x)-\xi_\uparrow(x_1))}dx_1=C_9e^{-C_8\xi_\uparrow(x)}\int\limits_0^{\xi_\uparrow(x)}\frac{e^{C_8\xi}d\xi}{(\xi_0+\xi)^2}=$$

$$C_9e^{-C_8\xi_\uparrow(x)}\Big(\int\limits_0^{\xi_\uparrow(x)/2}+\int\limits_{\xi_\uparrow(x)/2}^{\xi_\uparrow(x)}\Big)\frac{e^{C_8\xi}d\xi}{(\xi_0+\xi)^2}=C_9e^{-C_8\xi_\uparrow(x)}\Big[e^{C_8\xi_\uparrow(x)/2}\Big(\frac{1}{\xi_0}-\frac{1}{\xi_0+\xi_\uparrow(x)/2}\Big)+$$

$$\frac{1}{(\xi_0+\xi_\uparrow(x)/2)^2}\frac{1}{C_8}\Big(e^{C_8\xi_\uparrow(x)}-e^{C_8\xi_\uparrow(x)/2}\Big)\Big]\leqslant C_9\Big[\frac{1}{\xi_0}e^{-C_8\xi_\uparrow(x)/2}+\frac{4/C_8}{\xi^2_\uparrow(x)}\Big],$$

wherefrom Lemma 6 follows.

Lemma 7. *Let: 1) $V(x)\in\mathcal{K}$; 2) $\lambda\in\mathbb{C}$ and either $\lambda\leqslant 0$ or $\operatorname{Im}\lambda\neq 0$; 3) the integral*

$$\int\limits_{R_1}^{+\infty}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\qquad\Big(\int\limits_{-\infty}^{-R_1}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big)\tag{13}$$

converges. Then the integral

$$\int\limits_{R_0}^{+\infty}\frac{dx_1}{\sqrt{|V(x_1)|}}\qquad\Big(\int\limits_{-\infty}^{-R_0}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)\tag{14}$$

converges too,

$$\lim_{x\to\pm\infty}V(x)=\infty\tag{15}$$

and

$$\frac{1}{V(x)}=O\Big(\frac{1}{\xi_{\uparrow\downarrow}(x)}\int\limits_x^{\pm\infty}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big),\qquad x\to\pm\infty.\tag{16}$$

Proof.

As the integral (13) converges, then the integral

$$\int\limits_{R_1}^{+\infty}\operatorname{Re}\frac{1}{\sqrt{V(x_1)-\lambda}}dx_1\equiv\int\limits_{R_1}^{+\infty}\frac{\operatorname{Re}\sqrt{V(x_1)-\lambda}}{|V(x_1)-\lambda|}dx_1\qquad\Big(\int\limits_{-\infty}^{-R_1}\frac{\operatorname{Re}\sqrt{V(x_1)-\lambda}}{|V(x_1)-\lambda|}dx_1\Big)$$

converges too. In view of Lemma 5 there are constants C_1 and C_3 such that $\operatorname{Re}\sqrt{V(x)-\lambda}\geqslant C_3\sqrt{|V(x)|}$ and $|V(x)-\lambda|\leqslant\frac{1}{\sqrt{C_1}}|V(x)|$ for any $|x|\geqslant R_1$. Hence the integral

$$\int\limits_{R_1}^{+\infty}\frac{dx_1}{\sqrt{|V(x_1)|}}\leqslant\frac{1}{\sqrt{C_1}C_3}\int\limits_{R_1}^{+\infty}\frac{\operatorname{Re}\sqrt{V(x_1)-\lambda}}{|V(x_1)-\lambda|}dx_1$$

$$\Big(\int\limits_{-\infty}^{-R_1}\frac{dx_1}{\sqrt{|V(x_1)|}}\leqslant\frac{1}{\sqrt{C_1}C_3}\int\limits_{-\infty}^{-R_1}\frac{\operatorname{Re}\sqrt{V(x_1)-\lambda}}{|V(x_1)-\lambda|}dx_1\Big)$$

converges as much as the integral (14).

Let us now check (15) in the case $x \to +\infty$ (examination of the case $x \to -\infty$ is similar). The integral

$$\int\limits_{R_1}^{+\infty} \frac{V'(x_1)}{(V(x_1)-\lambda)^2} dx_1$$

converges owing to convergence of (14) and boundedness of $|V'|/|V-\lambda|^{3/2}$ for $x \geqslant R_1$ (see corollary 1). Hence the limit of the function

$$\frac{1}{V(x)-\lambda} = \frac{1}{V(R_1)-\lambda} - \int\limits_{R_1}^{x} \frac{V'(x_1)}{(V(x_1)-\lambda)^2} dx_1, \qquad x \geqslant R_1$$

for $x \to +\infty$ is finite. Moreover, because of convergence of (13) this limit is zero. Thus (15) holds.

Validity of (16) for $x \to +\infty$ (consideration of the case $x \to -\infty$ is similar) is justified by the fact that for $V \in \mathcal{K}$ there is $C_4 > 0$ such that $|V'|/|V|^{3/2} \leqslant C_4/\xi_\uparrow$ for any $x \geqslant R_0$ and by the chain

$$\frac{1}{|V(x)|} = \Big| \int\limits_{x}^{+\infty} \frac{V'(x_1)}{V^2(x_1)} dx_1 \Big| \leqslant C_4 \int\limits_{x}^{+\infty} \frac{dx_1}{\sqrt{|V(x_1)|}\xi_\uparrow(x_1)} \leqslant \frac{C_4}{\xi_\uparrow(x)} \int\limits_{x}^{+\infty} \frac{dx_1}{\sqrt{|V(x_1)|}}.$$

3. Asymptotics of formal eigenfunctions of a Hamiltonian

Asymptotic behavior of formal eigenfunctions of a Hamiltonian with potential belonging to $\mathcal{K}$ is described by the

Lemma 8. *Let: 1) $V(x) \in \mathcal{K}$; 2) $\lambda \in \mathbb{C}$ and either $\lambda \leqslant 0$ or $\operatorname{Im}\lambda \neq 0$. Then there are functions $\varphi_{0,\uparrow\downarrow}(x)$ normalizable at $\pm\infty$ being zero-modes of $h-\lambda$ and functions $\hat{\varphi}_{0,\uparrow\downarrow}(x)$ non-normalizable at $\pm\infty$ being zero-modes of $h-\lambda$ such that*[2]

$$\varphi_{0,\uparrow\downarrow}(x) = \frac{1}{\sqrt[4]{V(x)-\lambda}} e^{-\xi_{\uparrow\downarrow}(x;\lambda)} \Big[1 - \frac{1}{2} \int\limits_{x}^{\pm\infty} \alpha(x_1;\lambda)\, dx_1 + O\Big(\frac{1}{\xi^2_{\uparrow\downarrow}(x)}\Big)\Big], \qquad x \to \pm\infty, \quad (17)$$

$$\frac{\varphi'_{0,\uparrow\downarrow}(x)}{\varphi_{0,\uparrow\downarrow}(x)} = \mp\sqrt{V(x)-\lambda}\Big[1 \pm \frac{1}{4}\frac{V'(x)}{(V(x)-\lambda)^{3/2}} + O\Big(\frac{1}{\xi^2_{\uparrow\downarrow}(x)}\Big)\Big], \qquad x \to \pm\infty, \qquad (18)$$

$$\hat{\varphi}_{0,\uparrow\downarrow}(x) = \frac{1}{\sqrt[4]{V(x)-\lambda}} e^{\xi_{\uparrow\downarrow}(x;\lambda)} \Big[1 + \frac{1}{2} \int\limits_{x}^{\pm\infty} \alpha(x_1;\lambda)\, dx_1 + O\Big(\frac{1}{\xi^2_{\uparrow\downarrow}(x)}\Big)\Big], \qquad x \to \pm\infty, \quad (19)$$

$$\frac{\hat{\varphi}'_{0,\uparrow\downarrow}(x)}{\hat{\varphi}_{0,\uparrow\downarrow}(x)} = \pm\sqrt{V(x)-\lambda}\Big[1 \mp \frac{1}{4}\frac{V'(x)}{(V(x)-\lambda)^{3/2}} + O\Big(\frac{1}{\xi^2_{\uparrow\downarrow}(x)}\Big)\Big], \qquad x \to \pm\infty. \qquad (20)$$

Proof.

[2] Leading terms of asymptotics (17) and (19) are well known (see for example [11]).

We shall consider the case $x \to +\infty$ only because examination of the case $x \to -\infty$ is analogous. Let us show that the series

$$\varphi_{0,\uparrow}(x) = \frac{1}{\sqrt[4]{V(x)-\lambda}} \sum_{n=0}^{+\infty} \int_x^{+\infty} dx_1 \,\mathrm{sh}\,(\xi_\uparrow(x;\lambda) - \xi_\uparrow(x_1;\lambda))\alpha(x_1;\lambda)$$

$$\times \int_{x_1}^{+\infty} dx_2 \,\mathrm{sh}\,(\xi_\uparrow(x_1;\lambda) - \xi_\uparrow(x_2;\lambda))\alpha(x_2;\lambda)$$

$$\ldots \int_{x_{n-1}}^{+\infty} dx_n \,\mathrm{sh}\,(\xi_\uparrow(x_{n-1};\lambda) - \xi_\uparrow(x_n;\lambda))\alpha(x_n;\lambda)e^{-\xi_\uparrow(x_n\lambda)} \tag{21}$$

converges and gives the required function $\varphi_{0,\uparrow}(x)$. Convergence of (21) is provided by the fact that the series (21) is majorized by the series

$$\sum_{n=0}^{+\infty} \int_x^{+\infty} dx_1\, e^{\mathrm{Re}\,(\xi_\uparrow(x_1;\lambda)-\xi_\uparrow(x;\lambda))}\hat{\alpha}(x_1;\lambda) \int_{x_1}^{+\infty} dx_2\, e^{\mathrm{Re}\,(\xi_\uparrow(x_2;\lambda)-\xi_\uparrow(x_1;\lambda))}\hat{\alpha}(x_2;\lambda)\ldots$$

$$\int_{x_{n-1}}^{+\infty} dx_n\, e^{\mathrm{Re}\,(\xi_\uparrow(x_n;\lambda)-\xi_\uparrow(x_{n-1};\lambda))}\hat{\alpha}(x_n;\lambda)e^{-\mathrm{Re}\,\xi_\uparrow(x_n;\lambda)} \equiv$$

$$e^{-\mathrm{Re}\,\xi_\uparrow(x;\lambda)} \sum_{n=0}^{+\infty} \int_x^{+\infty} dx_1\, \hat{\alpha}(x_1;\lambda) \int_{x_1}^{+\infty} dx_2\, \hat{\alpha}(x_2;\lambda)\ldots \int_{x_{n-1}}^{+\infty} dx_n\, \hat{\alpha}(x_n;\lambda) =$$

$$e^{-\mathrm{Re}\,\xi_\uparrow(x;\lambda)} \sum_{n=0}^{+\infty} \frac{1}{n!}\Big(\int_x^{+\infty} \hat{\alpha}(x_1;\lambda), dx_1\Big)^n \equiv e^{-\mathrm{Re}\,\xi_\uparrow(x;\lambda)+I_{1,\uparrow}(x;\lambda)}.$$

From this estimate it follows also that due to Lemma 6 the asymptotics (17) for the function (21) is valid. Insofar as the series of first and second derivatives of (21) are majorized for x belonging to any segment $[x_1, x_2] \subset [R_1, +\infty[$ by the series independent of x,

$$\sum_{n=0}^{+\infty} \max_{[x_1,x_2]} |\xi'_\uparrow(x;\lambda)| \frac{1}{n!}(I_{1,\uparrow}(R_1;\lambda))^n$$

and

$$\sum_{n=0}^{+\infty} \Big\{ \max_{[x_1,x_2]} [|\xi''_\uparrow(x;\lambda)| + |\xi'_\uparrow(x;\lambda)|^2] \frac{1}{n!}(I_{1,\uparrow}(R_1;\lambda))^n +$$

$$\max_{[x_1,x_2]} [|\alpha(x;\lambda)\xi'_\uparrow(x;\lambda)|] \frac{1}{(n-1)!}(I_{1,\uparrow}(R_1;\lambda))^{n-1} \Big\},$$

it is possible to differentiate twice the series (21) term by term. Calculation of $h - \lambda$ applied to the function (21) allows to check that this function is a zero-mode of $h - \lambda$. One can check also (18), with the help of (21), using Lemma 6, corollary 1 and the fact that the absolute value of the derivative of n-th term in (21) is less than or equal to

$$\frac{1}{n!}\sqrt{|V(x)-\lambda|}(I_{1,\uparrow}(x;\lambda))^n e^{-\mathrm{Re}\,\xi_\uparrow(x;\lambda)}.$$

To prove normalizability of $\varphi_{0,\uparrow}(x)$ at $+\infty$ it is sufficient to prove normalizability at $+\infty$ of the leading term of the asymptotics (17). The latter comes out of the fact that for $V(x) \in \mathcal{K}$ the chain of inequalities holds due to Lemma 5,

$$\frac{e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}}{\sqrt{|V(x)-\lambda|}} \leqslant \frac{\sqrt[4]{C_2}}{\sqrt{|V(x)|}} e^{-2C_3 \int\limits_{R_1}^{x} \sqrt{|V(x_1)|}\,dx_1} \leqslant \frac{\sqrt[4]{C_2}}{\sqrt{\varepsilon}} e^{-2C_3 \int\limits_{R_1}^{x} \sqrt{\varepsilon}\,dx_1} \leqslant \frac{\sqrt[4]{C_2}}{\sqrt{\varepsilon}} e^{-2C_3\sqrt{\varepsilon}(x-R_1)},$$

the right side of which is obviously normalizable at $+\infty$.

Let us prove now that the required function $\hat{\varphi}_{0,\uparrow}(x)$ can be written in the form

$$\hat{\varphi}_{0,\uparrow}(x) = 2\varphi_{0,\uparrow}(x) \int\limits_{R_3}^{x} \frac{dx_1}{\varphi_{0,\uparrow}^2(x_1)}, \tag{22}$$

where $R_3 \geqslant R_1$ is a constant such that $\varphi_{0,\uparrow}(x)$ has no zeroes for $x \geqslant R_3$ (existence of R_3 is obvious because of (17)). The fact that the function (22) is a zero-mode of $h-\lambda$ follows from elementary calculations. To prove that the asymptotics (19) and (20) for the function (22) are valid it is sufficient to prove that

$$\int\limits_{R_3}^{x} \frac{dx_1}{\varphi_{0,\uparrow}^2(x_1)} = \frac{1}{2} e^{2\xi_\uparrow(x;\lambda)} \Big[1 + \int\limits_{x}^{+\infty} \alpha(x_1;\lambda)\,dx_1 + \Big(\frac{1}{\xi_\uparrow^2(x)}\Big)\Big]$$

in view of (17), (18), (22) and the obvious formula

$$\frac{\hat{\varphi}'_{0,\uparrow}(x)}{\hat{\varphi}_{0,\uparrow}(x)} = \frac{\varphi'_{0,\uparrow}(x)}{\varphi_{0,\uparrow}(x)} + \frac{1}{\varphi_{0,\uparrow}^2(x) \int\limits_{R_3}^{x} \frac{dx_1}{\varphi_{0,\uparrow}^2(x_1)}}.$$

By virtue of (17)

$$\frac{1}{\varphi_{0,\uparrow}^2(x)} = \sqrt{V(x)-\lambda}\; e^{2\xi_\uparrow(x;\lambda)} \Big[1 + \int\limits_{x}^{+\infty} \alpha(x_1;\lambda)\,dx_1 + \Big(\frac{1}{\xi_\uparrow^2(x)}\Big)\Big]. \tag{23}$$

Because of Lemma 6 the contribution of first and second terms of the right side of (23) at $\int_{R_3}^{x} dx_1/\varphi_{0,\uparrow}^2(x_1)$ is given by

$$\int\limits_{R_3}^{x} \sqrt{V(x_1)-\lambda}\; e^{2\xi_\uparrow(x_1;\lambda)} \Big[1 + \int\limits_{x_1}^{+\infty} \alpha(x_2;\lambda)\,dx_2\Big] dx_1 =$$

$$\frac{1}{2}\int\limits_{R_3}^{x} \Big(e^{2\xi_\uparrow(x_1;\lambda)}\Big)' \Big[1 + \int\limits_{x_1}^{+\infty} \alpha(x_2;\lambda)\,dx_2\Big] dx_1 =$$

$$\frac{1}{2}\Big\{e^{2\xi_\uparrow(x;\lambda)} \Big[1 + \int\limits_{x}^{+\infty} \alpha(x_2;\lambda)\,dx_2\Big] + O(1) + \int\limits_{R_3}^{x} e^{2\xi_\uparrow(x_1;\lambda)} \alpha(x_1;\lambda)\,dx_1\Big\} =$$

$$= \frac{1}{2} e^{2\xi_\uparrow(x;\lambda)} \Big[1 + \int\limits_{x}^{+\infty} \alpha(x_1;\lambda)\,dx_1 + \Big(\frac{1}{\xi_\uparrow^2(x)}\Big)\Big], \qquad x \to +\infty.$$

Due to local boundedness of $1/\varphi^2_{0,\uparrow}(x)$ for $x \geqslant R_3$ the contribution of the third term of right side of (23) is less than or equal to the integral

$$C_4 \int\limits_{R_3}^{x} \frac{\sqrt{|V(x_1)-\lambda|}}{(\xi_0+\xi_\uparrow(x_1))^2} e^{2\mathrm{Re}\,\xi_\uparrow(x_1;\lambda)} dx_1 \equiv$$

$$C_4 e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)} \int\limits_{R_3}^{x} \frac{\sqrt{|V(x_1)-\lambda|}}{(\xi_0+\xi_\uparrow(x_1))^2} e^{-2\int\limits_{x_1}^{x} \mathrm{Re}\,\sqrt{V(x_2)-\lambda}\,dx_2} dx_1, \tag{24}$$

where C_4 and ξ_0 are positive constants. For some positive constants C_5 and C_6 the integral (24), in view of Lemma 5, is less than or equal to the integral

$$C_5 e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)} \int\limits_{R_0}^{x} \frac{\sqrt{|V(x_1)|}}{(\xi_0+\xi_\uparrow(x_1))^2} e^{-C_6(\xi_\uparrow(x)-\xi_\uparrow(x_1))} dx_1 \equiv$$

$$C_5 e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)} \int\limits_{R_0}^{x} \frac{\xi'_\uparrow(x_1)}{(\xi_0+\xi_\uparrow(x_1))^2} e^{-C_6(\xi_\uparrow(x)-\xi_\uparrow(x_1))} dx_1,$$

which is equal (see the proof of Lemma 6) to $O(e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}/\xi^2_\uparrow(x))$, $x \to +\infty$. Thus, (19) and (20) hold.

Finally in order to prove non-normalizability at $+\infty$ of function (22) let us first prove the auxiliary inequality

$$|V(x)| \leqslant C_0(\xi_0+\xi_\uparrow(x))^\gamma, \qquad x \geqslant R_0, \tag{25}$$

where C_0 and γ are some positive constants. This equality for $V(x) \in \mathcal{K}$ follows from the sequence

$$|V(x)| = |V(R_0)e^{\ln(V(x)/V(R_0))}| \leqslant |V(R_0)|e^{\int\limits_{R_0}^{x} \frac{|V'(x_1)|}{|V(x_1)|}dx_1} \leqslant$$

$$|V(R_0)|e^{\gamma\int\limits_{R_0}^{x} \frac{\sqrt{|V(x_1)|}}{\xi_0+\xi_\uparrow(x_1)}dx_1} \equiv |V(R_0)|e^{\gamma\int\limits_{R_0}^{x} \frac{\xi'_\uparrow(x_1)}{\xi_0+\xi_\uparrow(x_1)}dx_1} = |V(R_0)|\frac{(\xi_0+\xi_\uparrow(x))^\gamma}{(\xi_0+\xi_\uparrow(R_0))^\gamma}.$$

To prove non-normalizability at $+\infty$ of function (22) it is sufficient to prove non-normalizability at $+\infty$ of the leading term of asymptotics (19). The latter is provided by the fact that in view of (25) and Lemma 5 the chain holds,

$$\frac{e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}}{\sqrt{|V(x)-\lambda|}} \geqslant \frac{\sqrt[4]{C_1}}{\sqrt{|V(x)|}} e^{2C_3\int\limits_{R_1}^{x}\sqrt{|V(x_1)|}\,dx_1} \equiv \frac{\sqrt[4]{C_1}\xi'_\uparrow(x_1)}{|V(x)|} e^{2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_0))} \geqslant$$

$$\frac{\sqrt[4]{C_2}}{C_0}\frac{\xi'_\uparrow(x)}{(\xi_0+\xi_\uparrow(x))^\gamma} e^{2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_0))},$$

the right side of which is non-normalizable at $+\infty$. Lemma 8 is proved.

Remark 1. The asymptotics (19) and (20) are valid for any zero-mode of $h-\lambda$ linear independent of $\varphi_{0,\uparrow\downarrow}(x)$ (after its proper normalization).

Corollary 2. Let Hamiltonians $h^\pm = -\partial^2 + V_{1,2}(x)$ be intertwined by $q_1^\pm = \mp\partial + \chi(x)$:

$$q_1^\pm h^\mp = h^\pm q_1^\pm, \qquad h^\pm = q_1^\pm q_1^\mp + \lambda = \lambda + \chi^2 \mp \chi'.$$

Suppose also $\varphi(x)$ to be a zero-mode of q_1^- so that $\chi = -\varphi'/\varphi$. Then

$$\triangle V \equiv V_2 - V_1 = 2\chi' \equiv -2(\ln\varphi)'' \equiv -2\Big[\frac{\varphi''}{\varphi} - \Big(\frac{\varphi'}{\varphi}\Big)^2\Big] \equiv -2\Big[V_1 - \lambda - \Big(\frac{\varphi'}{\varphi}\Big)^2\Big].$$

At last suppose that $V_1(x) \in \mathcal{K}$, $\lambda \in \mathbb{C}$ and either $\lambda \leqslant 0$ or $\operatorname{Im}\lambda \neq 0$. Then because of (18), (20), Lemma 5 and Corollary 1

$$\triangle V = \pm\frac{V_1'(x)}{\sqrt{V_1(x) - \lambda}} + O\Big(\frac{V_1(x)}{\xi_\uparrow^2(x)}\Big), \qquad x \to +\infty, \tag{26}$$

if at $x \to +\infty$ the asymptotics (17) or respectively (19) is valid for φ. For the case when at $x \to -\infty$ the asymptotics (17) or respectively (19) is valid for φ,

$$\triangle V = \mp\frac{V_1'(x)}{\sqrt{V_1(x) - \lambda}} + O\Big(\frac{V_1(x)}{\xi_\downarrow^2(x)}\Big), \qquad x \to -\infty. \tag{27}$$

Finally,

$$\triangle V' \equiv -2\Big\{V_1' - 2\frac{\varphi'}{\varphi}\Big[\frac{\varphi''}{\varphi} - \Big(\frac{\varphi'}{\varphi}\Big)^2\Big]\Big\} \equiv -2\Big[V_1' + \frac{\varphi'}{\varphi}\triangle V\Big] = O\Big(\frac{V_1^{3/2}(x)}{\xi_{\uparrow\downarrow}^2(x)}\Big), \qquad x \to \pm\infty, \tag{28}$$

$$\triangle V'' \equiv -2\Big\{V_1'' + \triangle V\Big[\frac{\varphi''}{\varphi} - \Big(\frac{\varphi'}{\varphi}\Big)^2\Big] + \frac{\varphi'}{\varphi}\triangle V'\Big\} \equiv$$

$$-2\Big[V_1'' - \frac{1}{2}\triangle V^2 + \frac{\varphi'}{\varphi}\triangle V'\Big] = O\Big(\frac{V_1^2(x)}{\xi_{\uparrow\downarrow}^2(x)}\Big), \qquad x \to \pm\infty, \tag{29}$$

independently of asymptotics of φ.

It also follows from (26), (27), Lemma 5 and Corollary 1 that

$$\triangle V = O\Big(\frac{V_1(x) - \lambda}{\xi_\uparrow(x)}\Big) = o(V_1 - \lambda) = o(V_1), \qquad x \to \pm\infty,$$

i.e. that

$$V_2 - \lambda = V_1 - \lambda + o(V_1 - \lambda) = (V_1 - \lambda)[1 + o(1)], \qquad x \to \pm\infty, \tag{30}$$

$$V_2 = V_1 + o(V_1) = V_1[1 + o(1)], \qquad x \to \pm\infty. \tag{31}$$

Corollary 3 (1). There are no degenerate eigenvalues of the Hamiltonian with a potential belonging to K, satisfying either $\lambda \leqslant 0$ or $\operatorname{Im}\lambda \neq 0$, *i.e.* eigenvalues, whose geometric multiplicity is more than 1 (eigenvalues, for which there are more than one linearly independent eigenfunction). Hence, for the Hamiltonian with a potential belonging to K there are no more than one Jordan cell made of an eigenfunction and associated functions, normalizable on the whole axis, for any given eigenvalue λ such that either $\lambda \leqslant 0$ or $\operatorname{Im}\lambda \neq 0$.

4. Asymptotics of formal associated functions of a Hamiltonian

The asymptotic behavior of formal associated functions of a Hamiltonian h with a potential belonging to $\mathcal{K}$ is characterized by

Lemma 9. *Let: 1)* $h = -\partial^2 + V(x)$, $V(x) \in \mathcal{K}$*; 2)* $\lambda \in \mathbb{C}$ *and either* $\lambda \leqslant 0$ *or* $\operatorname{Im}\lambda \neq 0$*; 3)* $\eta_{\uparrow\downarrow}(x) = \pm \int\limits_{\pm R_0}^{x} dx_1/\sqrt{|V(x_1)|}$. *Then there are denumerable sequences:* $\varphi_{n,\uparrow\downarrow}(x)$ *of formal associated functions of* h *for a spectral value* λ*, normalizable at* $\pm\infty$*, and* $\hat{\varphi}_{n,\uparrow\downarrow}(x)$ *of formal associated functions, non-normalizable at* $\pm\infty$*, such that:*

$$h\varphi_{0,\uparrow\downarrow} = \lambda\varphi_{0,\uparrow\downarrow}, \qquad (h-\lambda)\varphi_{n,\uparrow\downarrow} = \varphi_{n-1,\uparrow\downarrow}, \quad n \geqslant 1, \tag{32}$$

$$h\hat{\varphi}_{0,\uparrow\downarrow} = \lambda\hat{\varphi}_{0,\uparrow\downarrow}, \qquad (h-\lambda)\hat{\varphi}_{n,\uparrow\downarrow} = \hat{\varphi}_{n-1,\uparrow\downarrow}, \quad n \geqslant 1; \tag{33}$$

if $\pm \int\limits_{\pm R_0}^{\pm\infty} dx_1/\sqrt{|V(x_1)|} < +\infty$*, then for* $x \to \pm\infty$*,*

$$\varphi_{n,\uparrow\downarrow}(x) = \frac{1}{n!\,\sqrt[4]{V(x)-\lambda}}\left(\pm\frac{1}{2}\int\limits_{\pm\infty}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\right)^n e^{-\xi_{\uparrow\downarrow}(x;\lambda)}\left[1+O\left(\frac{1}{\xi_{\uparrow\downarrow}(x)}\right)\right], \tag{34}$$

$$\hat{\varphi}_{n,\uparrow\downarrow}(x) = \frac{1}{n!\,\sqrt[4]{V(x)-\lambda}}\left(\mp\frac{1}{2}\int\limits_{\pm\infty}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\right)^n e^{\xi_{\uparrow\downarrow}(x;\lambda)}\left[1+O\left(\frac{1}{\xi_{\uparrow\downarrow}(x)}\right)\right], \tag{35}$$

$$\varphi'_{n,\uparrow\downarrow}(x) = \mp\frac{1}{n!}\sqrt[4]{V(x)-\lambda}\left(\pm\frac{1}{2}\int\limits_{\pm\infty}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\right)^n e^{-\xi_{\uparrow\downarrow}(x;\lambda)}\left[1+O\left(\frac{1}{\xi_{\uparrow\downarrow}(x)}\right)\right] \tag{36}$$

and if $\pm \int\limits_{\pm R_0}^{\pm\infty} dx_1/\sqrt{|V(x_1)|} = +\infty$*, then*

$$\varphi_{n,\uparrow\downarrow}(x) = \frac{1}{n!\,\sqrt[4]{V(x)-\lambda}}\left(\pm\frac{1}{2}\int\limits_{\pm R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\right)^n e^{-\xi_{\uparrow\downarrow}(x;\lambda)}\left[1+O\left(\frac{\ln\eta_{\uparrow\downarrow}(x)}{\eta_{\uparrow\downarrow}(x)}\right)\right], \tag{37}$$

$$\hat{\varphi}_{n,\uparrow\downarrow}(x) = \frac{1}{n!\,\sqrt[4]{V(x)-\lambda}}\left(\mp\frac{1}{2}\int\limits_{\pm R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\right)^n e^{\xi_{\uparrow\downarrow}(x;\lambda)}\left[1+O\left(\frac{\ln\eta_{\uparrow\downarrow}(x)}{\eta_{\uparrow\downarrow}(x)}\right)\right], \tag{38}$$

$$\varphi'_{n,\uparrow\downarrow}(x) = \mp\frac{1}{n!}\sqrt[4]{V(x)-\lambda}\left(\pm\frac{1}{2}\int\limits_{\pm R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\right)^n e^{-\xi_{\uparrow\downarrow}(x;\lambda)}\left[1+O\left(\frac{\ln\eta_{\uparrow\downarrow}(x)}{\eta_{\uparrow\downarrow}(x)}\right)\right]. \tag{39}$$

Proof.

Let us prove the existence of $\varphi_{n,\uparrow}(x)$ and $\hat{\varphi}_{n,\uparrow}(x)$ only, because the proof of existence of $\varphi_{n,\downarrow}(x)$ and $\hat{\varphi}_{n,\downarrow}(x)$ is analogous. The existence of $\varphi_{0,\uparrow}(x)$ and $\hat{\varphi}_{0,\uparrow}(x)$ was proved in

Lemma 8 and in view of $V(x) \in \mathcal{K}$ the estimate $O(1/\xi_\uparrow(x)) = O(\ln \eta_\uparrow(x)/\eta_\uparrow(x))$, $x \to +\infty$ (cf. Lemma 9 and Lemma 8) follows from the chain

$$\eta_\uparrow(x) = \int\limits_{R_0}^{x} \frac{dx_1}{\sqrt{|V(x_1)|}} \leqslant \frac{1}{\varepsilon} \int\limits_{R_0}^{x} \sqrt{|V(x_1)|}\, dx_1 = \frac{1}{\varepsilon} \xi_\uparrow(x). \tag{40}$$

Suppose now the existence of $\varphi_{l,\uparrow}(x)$ and $\hat{\varphi}_{l,\uparrow}(x)$ and let us prove the existence of $\varphi_{l+1,\uparrow}(x)$ and $\hat{\varphi}_{l+1,\uparrow}(x)$. In this way the Lemma will be completely proved.

Consider the case $\int\limits_{R_0}^{+\infty} dx_1/\sqrt{|V(x_1)|} < +\infty$. One can check that in this case $\varphi_{l+1,\uparrow}(x)$ and $\hat{\varphi}_{l+1,\uparrow}(x)$ can be written in the form

$$\varphi_{l+1,\uparrow}(x) = -\frac{1}{2}\Big\{\hat{\varphi}_{0,\uparrow}(x) \int\limits_{+\infty}^{x} \varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1 - \varphi_{0,\uparrow}(x) \int\limits_{+\infty}^{x} \hat{\varphi}_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1\Big\}, \tag{41}$$

$$\hat{\varphi}_{l+1,\uparrow}(x) = -\frac{1}{2}\Big\{\hat{\varphi}_{0,\uparrow}(x) \int\limits_{+\infty}^{x} \varphi_{0,\uparrow}(x_1)\hat{\varphi}_{l,\uparrow}(x_1)\, dx_1 - \varphi_{0,\uparrow}(x) \int\limits_{R_1}^{x} \hat{\varphi}_{0,\uparrow}(x_1)\hat{\varphi}_{l,\uparrow}(x_1)\, dx_1\Big\}. \tag{42}$$

Convergence of $\int\limits_{+\infty}^{x} \hat{\varphi}_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1$ follows from the fact that due to (34), (35) and Lemma 5 there is constant $C_4 > 0$ such that

$$\Big|\int\limits_{+\infty}^{x} \hat{\varphi}_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1\Big| \leqslant C_4 \int\limits_{x}^{+\infty} \frac{dx_1}{\sqrt{|V(x_1)|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l =$$

$$\frac{C_4}{l+1}\Big(\int\limits_{x}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l+1} < +\infty.$$

Convergence of $\int\limits_{+\infty}^{x} \varphi_{0,\uparrow}(x_1)\hat{\varphi}_{l,\uparrow}(x_1)\, dx_1$ can be proved analogously. Convergence of the integral $\int\limits_{+\infty}^{x} \varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1$ is obvious. Thus the right sides of (41) and (42) are well defined. The fact that the right sides of (41) and (42) satisfy (32) and (33) for $n = l+1$ can be checked by direct application of h to these sides. One must take into account here that the Wronskian $\hat{\varphi}'_{0,\uparrow}(x)\varphi_{0,\uparrow}(x) - \hat{\varphi}_{0,\uparrow}(x)\varphi'_{0,\uparrow}(x) \equiv 2$, that follows from the asymptotics of Lemma 8.

Now we transform the integrals in (41) and (42). In view of (34) and (35) the integrand of $\int\limits_{+\infty}^{x}\hat{\varphi}_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1$ reads

$$\hat{\varphi}_{0,\uparrow}(x)\varphi_{l,\uparrow}(x) = \frac{1}{2^l l! \sqrt{V(x) - \lambda}}\Big(\int\limits_{+\infty}^{x} \frac{dx_1}{\sqrt{V(x_1) - \lambda}}\Big)^l \Big[1 + \Big(\frac{1}{\xi_\uparrow(x)}\Big)\Big], \qquad x \to +\infty. \tag{43}$$

The first term of the right side of (43) in the integral is equal to

$$\frac{1}{2^l l!} \int\limits_{+\infty}^{x} \frac{dx_1}{\sqrt{V(x_1) - \lambda}} \Big(\int\limits_{+\infty}^{x_1} \frac{dx_2}{\sqrt{V(x_2) - \lambda}}\Big)^l = \frac{2}{(l+1)!}\Big(\frac{1}{2}\int\limits_{+\infty}^{x} \frac{dx_1}{\sqrt{V(x_1) - \lambda}}\Big)^{l+1}, \tag{44}$$

and the absolute value of contribution of the second term is less than or equal to

$$C_5 \int\limits_x^{+\infty} \frac{dx_1}{\sqrt{|V(x_1)-\lambda|}} \frac{1}{\xi_\uparrow(x_1)} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)-\lambda|}} \Big)^l,$$

for some constant $C_5 > 0$. From Lemma 5 the latter expression can be estimated in the following way,

$$\frac{C_5}{\xi_\uparrow(x)} \int\limits_x^{+\infty} \frac{dx_1}{\sqrt{|V(x_1)-\lambda|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)-\lambda|}} \Big)^l = \frac{C_5}{(l+1)\xi_\uparrow(x)} \Big(\int\limits_x^{+\infty} \frac{dx_1}{\sqrt{|V(x_1)-\lambda|}} \Big)^{l+1} \leqslant$$

$$\frac{C_6}{\xi_\uparrow(x)} \Big(\int\limits_x^{+\infty} \frac{\operatorname{Re}\sqrt{V^*(x_1)-\lambda^*}}{|V(x_1)-\lambda|} dx_1 \Big)^{l+1} \leqslant \frac{C_6}{\xi_\uparrow(x)} \Big| \int\limits_x^{+\infty} \frac{\sqrt{V^*(x_1)-\lambda^*}}{|V(x_1)-\lambda|} dx_1 \Big|^{l+1} =$$

$$\frac{C_6}{\xi_\uparrow(x)} \Big| \int\limits_x^{+\infty} \frac{dx_1}{\sqrt{V(x_1)-\lambda}} \Big|^{l+1}, \tag{45}$$

for some constant $C_6 > 0$. Thus

$$\int\limits_{+\infty}^x \hat{\varphi}_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1 = \frac{2}{(l+1)!} \Big(\frac{1}{2} \int\limits_{+\infty}^x \frac{dx_1}{\sqrt{V(x_1)-\lambda}} \Big)^{l+1} \Big[1 + \Big(\frac{1}{\xi_\uparrow(x)} \Big) \Big], \qquad x \to +\infty. \tag{46}$$

In view of (34), $V(x) \in \mathcal{K}$, Lemmas 5 and 7 the following estimate for the integral $\int\limits_{+\infty}^{x} \varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1$ is valid[3],

$$\Big| \int\limits_{+\infty}^x \varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\, dx_1 \Big| \leqslant C_7 \int\limits_x^{+\infty} \frac{dx_1}{\sqrt{|V(x_1)-\lambda|}} e^{-2\operatorname{Re}\xi_\uparrow(x_1;\lambda)} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)-\lambda|}} \Big)^l \leqslant$$

$$C_8 e^{-2\operatorname{Re}\xi_\uparrow(x;\lambda)} \int\limits_x^{+\infty} \frac{e^{-2\int_x^{x_1} \operatorname{Re}\sqrt{V(x_3)-\lambda}\, dx_3}}{\sqrt{|V(x_1)|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l dx_1 \leqslant$$

$$C_8 e^{-2\operatorname{Re}\xi_\uparrow(x;\lambda)} \int\limits_x^{+\infty} \frac{e^{-C_9\int_x^{x_1} \sqrt{|V(x_3)|}\, dx_3}}{\sqrt{|V(x_1)|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l dx_1 =$$

$$C_8 e^{-2\operatorname{Re}\xi_\uparrow(x;\lambda)+C_9\xi_\uparrow(x)} \int\limits_x^{+\infty} \frac{e^{-C_9\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l dx_1 =$$

$$\frac{C_8}{C_9} e^{-2\operatorname{Re}\xi_\uparrow(x;\lambda)+C_9\xi_\uparrow(x)} \Bigg\{ \frac{e^{-C_9\xi_\uparrow(x)}}{|V(x)|} \Big(\int\limits_x^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l +$$

[3] The latter equality in (47) is obtained with the help of the same trick as in (45)

$$\int_x^{+\infty} e^{-C_9\xi_\uparrow(x_1)}\Big[-\frac{l}{|V(x_1)|^{3/2}}\Big(\int_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l-1}-$$

$$\frac{\operatorname{Re}V(x_1)\operatorname{Re}V'(x_1)+\operatorname{Im}V(x_1)\operatorname{Im}V'(x_1)}{|V(x_1)|^3}\Big(\int_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l}\Big]dx_1\Big\}\leqslant$$

$$\frac{C_8}{C_9}e^{-2\operatorname{Re}\xi_\uparrow(x;\lambda)}\Big[\frac{1}{|V(x)|}\Big(\int_x^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l}+$$

$$C_{10}e^{C_9\xi_\uparrow(x)}\int_x^{+\infty}\frac{e^{-C_9\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\frac{1}{\xi_\uparrow(x_1)}\Big(\int_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l}dx_1\Big]=$$

$$O\Big(e^{-2\xi_\uparrow(x;\lambda)}\frac{1}{V(x)}\Big(\int_x^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l}\Big)+O\Big(e^{-2\xi_\uparrow(x;\lambda)}\frac{1}{\xi_\uparrow(x)}\Big(\int_x^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l+1}\Big)=$$

$$O\Big(\frac{e^{-2\xi_\uparrow(x;\lambda)}}{\xi_\uparrow(x)}\Big(\int_x^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)-\lambda|}}\Big)^{l+1}\Big)=O\Big(\frac{e^{-2\xi_\uparrow(x;\lambda)}}{\xi_\uparrow(x)}\Big(\int_x^{+\infty}\frac{dx_2}{\sqrt{V(x_2)-\lambda}}\Big)^{l+1}\Big),\qquad(47)$$

for some positive constants $C_7,\dots C_{10}$. The asymptotics (34) and (36) for $n=l+1$ follow from (41), (46), (47), from (34) and (35) with $n=0$ and from (18), (20) and Corollary 1.

The integral $\int\limits_{+\infty}^{x}\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1$ can be calculated in the same way and the result is

$$\int_{+\infty}^{x}\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1=\frac{-2}{(l+1)!}\Big(-\frac{1}{2}\int_{+\infty}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big)^{l+1}\Big[1+O\Big(\frac{1}{\xi_\uparrow(x)}\Big)\Big],\quad x\to+\infty.\qquad(48)$$

One can also obtain the estimate,

$$\int_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1=O\Big(\frac{e^{2\xi_\uparrow(x;\lambda)}}{\xi_\uparrow(x)}\Big(\int_{+\infty}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big)^{l+1}\Big),\qquad x\to+\infty.\qquad(49)$$

Then in view of (34), (35) for $n=0$ and (42), (48), (49) the asymptotics (35) turns out to be valid for $n=l+1$. For the integral $\int\limits_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1$ the following estimate can be derived for some positive constants $C_{11},\dots,C_{15}$, ξ_0,

$$\Big|\int_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1\Big|\leqslant C_{11}\int_{R_1}^{x}\frac{e^{2\operatorname{Re}\xi_\uparrow(x_1;\lambda)}}{\sqrt{|V(x_1)-\lambda|}}\Big(\int_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)-\lambda|}}\Big)^{l}dx_1\leqslant$$

$$C_{12}e^{2\operatorname{Re}\xi_\uparrow(x;\lambda)}\int_{R_0}^{x}\frac{e^{-C_{13}(\xi_\uparrow(x)-\xi_\uparrow(x_1))}}{\sqrt{|V(x_1)|}}\Big(\int_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l}dx_1=$$

$$\frac{C_{12}}{C_{13}} e^{2\mathrm{Re}\,\xi_{\uparrow}(x;\lambda)-C_{13}\xi_{\uparrow}(x)} \Big\{ \frac{e^{C_{13}\xi_{\uparrow}(x_1)}}{|V(x_1)|} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l \Big|_{R_0}^{x} -$$

$$\int\limits_{R_0}^{x} e^{C_{13}\xi_{\uparrow}(x_1)} \Big[-\frac{l}{|V(x_1)|^{3/2}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^{l-1}$$

$$-\frac{\mathrm{Re}\,V(x_1)\mathrm{Re}\,V'(x_1) + \mathrm{Im}\,V(x_1)\mathrm{Im}\,V'(x_1)}{|V(x_1)|^3} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l \Big] dx_1 \Big\} \leqslant$$

$$\frac{C_{12}}{C_{13}} e^{2\mathrm{Re}\,\xi_{\uparrow}(x;\lambda)} \Big[\frac{1}{|V(x)|} \Big(\int\limits_{x}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l +$$

$$C_{14} e^{-C_{13}\xi_{\uparrow}(x)} \int\limits_{R_0}^{x} \frac{e^{C_{13}\xi_{\uparrow}(x_1)}}{\sqrt{|V(x_1)|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l \frac{dx_1}{\xi_0 + \xi_{\uparrow}(x_1)} \Big] \leqslant$$

$$\frac{C_{12}}{C_{13}} e^{2\mathrm{Re}\,\xi_{\uparrow}(x;\lambda)} \Big[\frac{C_{15}}{\xi_0 + \xi_{\uparrow}(x)} \Big(\int\limits_{x}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^{l+1} +$$

$$C_{14} e^{-C_{13}\xi_{\uparrow}(x)} \int\limits_{R_0}^{x} \frac{e^{C_{13}\xi_{\uparrow}(x_1)}}{\sqrt{|V(x_1)|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l \frac{dx_1}{\xi_0 + \xi_{\uparrow}(x_1)} \Big], \tag{50}$$

with the help of (35) for $n = 0$ and $n = l$, $V(x) \in \mathcal{K}$ and Lemmas 5 and 7. Let us show that

$$e^{-C_{13}\xi_{\uparrow}(x)} \int\limits_{R_0}^{x} \frac{e^{C_{13}\xi_{\uparrow}(x_1)}}{\sqrt{|V(x_1)|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l \frac{dx_1}{\xi_0 + \xi_{\uparrow}(x_1)} =$$

$$o\Big(\frac{1}{\xi_{\uparrow}(x)} \Big(\int\limits_{x}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^{l+1} \Big), \qquad x \to +\infty. \tag{51}$$

Then using (50) and the fact that in accordance to Lemma 5

$$\int\limits_{x}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} = O\Big(\int\limits_{x}^{+\infty} \frac{\mathrm{Re}\,\sqrt{V^*(x_2) - \lambda^*}}{|V(x_2) - \lambda|} dx_2 \Big) =$$

$$O\Big(\int\limits_{x}^{+\infty} \frac{\sqrt{V^*(x_2) - \lambda^*}}{|V(x_2) - \lambda|} dx_2 \Big) = O\Big(\int\limits_{x}^{+\infty} \frac{dx_2}{\sqrt{V(x_2) - \lambda}} \Big), \qquad x \to +\infty,$$

the required estimate (49) would be proved.

Performing the change of variable $\xi = \xi_{\uparrow}(x_2)$, we get

$$e^{-C_{13}\xi_{\uparrow}(x)} \int\limits_{R_0}^{x} \frac{e^{C_{13}\xi_{\uparrow}(x_1)}}{\sqrt{|V(x_1)|}} \Big(\int\limits_{x_1}^{+\infty} \frac{dx_2}{\sqrt{|V(x_2)|}} \Big)^l \frac{dx_1}{\xi_0 + \xi_{\uparrow}(x_1)} =$$

$$e^{-C_{13}\xi_\uparrow(x)}\Big(\int\limits_0^{\xi_\uparrow(x)/2}+\int\limits_{\xi_\uparrow(x)/2}^{\xi_\uparrow(x)}\Big)\frac{e^{C_{13}\xi}}{|V(x_1(\xi))|}\Big(\int\limits_{x_1(\xi)}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l\frac{d\xi}{\xi_0+\xi}\leqslant$$

$$e^{-C_{13}\xi_\uparrow(x)}\Bigg\{\frac{1}{\xi_0}e^{C_{13}\xi_\uparrow(x)/2}\int\limits_0^{\xi_\uparrow(x)/2}\Big(\int\limits_{x_1(\xi)}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l\frac{d\xi}{|V(x_1(\xi))|}+$$

$$\frac{1}{\xi_0+\xi_\uparrow(x)/2}\int\limits_{\xi_\uparrow(x)/2}^{\xi_\uparrow(x)}\frac{e^{C_{13}\xi}}{|V(x_1(\xi))|}\Big(\int\limits_{x_1(\xi)}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l d\xi\Bigg\}\leqslant$$

$$\frac{e^{-C_{13}\xi_\uparrow(x)/2}}{\xi_0}\int\limits_{R_0}^{+\infty}\Big(\int\limits_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l\frac{dx_1}{\sqrt{|V(x_1)|}}+\frac{2e^{-C_{13}\xi_\uparrow(x)}}{\xi_\uparrow(x)}\int\limits_{R_0}^{x}\frac{e^{C_{13}\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\Big(\int\limits_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l dx_1=$$

$$\frac{e^{-C_{13}\xi_\uparrow(x)/2}}{(l+1)\xi_0}\Big(\int\limits_{R_0}^{+\infty}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{l+1}+\frac{2e^{-C_{13}\xi_\uparrow(x)}}{\xi_\uparrow(x)}\int\limits_{R_0}^{x}\frac{e^{C_{13}\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\Big(\int\limits_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l dx_1. \quad (52)$$

It follows from (52) and from the estimate of $\int\limits_{R_0}^{x}\frac{e^{C_{13}\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\Big(\int\limits_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l dx_1$ contained in (50), that

$$e^{-C_{13}\xi_\uparrow(x)}\int\limits_{R_0}^{x}\frac{e^{C_{13}\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\Big(\int\limits_{x_1}^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l\frac{dx_1}{\xi_0+\xi_\uparrow(x_1)}=$$

$$O\big(e^{-C_{13}\xi_\uparrow(x)/2}\big)+o\Big(\frac{1}{\xi_\uparrow(x)}\Big(\int\limits_x^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l+1}\Big). \quad (53)$$

As well it follows from (16) and (25) that

$$O\big(e^{-C_{13}\xi_\uparrow(x)/2}\big)=o\big(\xi_\uparrow^{l-(l+1)\gamma}(x)\big)=o\Big(\frac{\xi_\uparrow^l(x)}{|V(x)|^{l+1}}\Big)=$$

$$o\Big(\frac{1}{\xi_\uparrow(x)}\Big(\int\limits_x^{+\infty}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l+1}\Big),\qquad x\to+\infty. \quad (54)$$

Then the estimate (51) is derived from (54) and (53). Thus, (35) is valid for $n=l+1$.

Finally let us show that functions $\varphi_{n,\uparrow}(x)$ ($\hat\varphi_{n,\uparrow}(x)$) for any n are normalizable (non-normalizable) at $+\infty$. For this purpose it is sufficient to prove that the leading term of (34) ((35)) is normalizable (non-normalizable) at $+\infty$. Normalizability of the leading term of (34) is owed to the fact that, because of Lemma 5, the following estimates take place,

$$\frac{1}{\sqrt{|V(x)-\lambda|}}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big|\int\limits_{+\infty}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big|^{2n}\leqslant$$

$$\frac{C_2^{(2n+1)/4}}{\sqrt{|V(x)|}}e^{-2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))}\Big(\int\limits_x^{+\infty}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{2n}\leqslant$$

$$C_2^{(2n+1)/4}\frac{\xi'_\uparrow(x)}{|V(x)|}e^{-2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))}\Big(\int\limits_{R_0}^{+\infty}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{2n}\leqslant$$

$$\frac{C_2^{(2n+1)/4}}{\varepsilon}\Big(\int\limits_{R_0}^{+\infty}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{2n}\xi'_\uparrow(x)e^{-2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))},\qquad x\geqslant R_1,$$

where the latter expression is normalizable at $+\infty$. Non-normalizability of the leading term (35) follows from the fact that in view of Lemma 5 and (16), (25) the estimates are valid for some constant $C_{16}>0$,

$$\frac{1}{\sqrt{|V(x)-\lambda|}}e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big|\int\limits_{+\infty}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big|^{2n}\geqslant$$

$$\frac{\sqrt[4]{C_1}}{\sqrt{|V(x)|}}e^{2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))}\Big(\int\limits_{x}^{+\infty}\frac{\mathrm{Re}\,\sqrt{V^*(x_1)-\lambda^*}}{|V(x_1)-\lambda|}dx_1\Big)^{2n}\geqslant$$

$$\frac{C_1^{(4n+1)/4}C_3^{2n}}{|V(x)|}\xi'_\uparrow(x)e^{2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))}\Big(\int\limits_{x}^{+\infty}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{2n}\geqslant$$

$$C_{16}\frac{\xi'_\uparrow(x)\xi_\uparrow^{2n}(x)}{|V(x)|^{2n+1}}e^{2C_3\xi_\uparrow(x)}\geqslant\frac{C_{16}}{C_0^{2n+1}}\frac{\xi'_\uparrow(x)\xi_\uparrow^{2n}(x)}{(\xi_0+\xi_\uparrow(x))^{(2n+1)\gamma}}e^{2C_3\xi_\uparrow(x)},\qquad x\geqslant R_1,\tag{55}$$

where the latter expression is non-normalizable at $+\infty$.

Let us now consider the case $\int\limits_{R_0}^{+\infty}dx_1/\sqrt{|V(x_1)|}=+\infty$ and prove that $\varphi_{l+1,\uparrow}(x)$ and $\hat\varphi_{l+1,\uparrow}(x)$ can be written in the form

$$\varphi_{l+1,\uparrow}(x)=-\frac{1}{2}\Big\{\hat\varphi_{0,\uparrow}(x)\int\limits_{+\infty}^{x}\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1-\varphi_{0,\uparrow}(x)\int\limits_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1\Big\},\tag{56}$$

$$\hat\varphi_{l+1,\uparrow}(x)=-\frac{1}{2}\Big\{\hat\varphi_{0,\uparrow}(x)\int\limits_{R_1}^{x}\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1-\varphi_{0,\uparrow}(x)\int\limits_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1\Big\}.\tag{57}$$

Convergence of $\int\limits_{+\infty}^{x}\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1$ follows from the fact that $V(x)\in\mathcal{K}$ and in view of Lemma 5, and (37) for $n=0$ and $n=l$ there are positive constants C_3, C_{17}, ε such that

$$\Big|\int\limits_{+\infty}^{x}\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1\Big|\leqslant C_{17}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)+2C_3\xi_\uparrow(x)}\int\limits_{x}^{+\infty}\frac{e^{-2C_3\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\Big(\int\limits_{R_1}^{x_1}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l}dx_1\leqslant$$

$$\frac{C_{17}}{\varepsilon^l}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)+2C_3\xi_\uparrow(x)}\int\limits_{x}^{+\infty}\frac{\xi_\uparrow^l(x_1)}{\sqrt{|V(x_1)|}}e^{-2C_3\xi_\uparrow(x_1)}dx_1\leqslant$$

$$\frac{C_{17}}{\varepsilon^{l+1}}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)+2C_3\xi_\uparrow(x)}\int\limits_x^{+\infty}\xi_\uparrow^l(x_1)\xi_\uparrow'(x_1)e^{-2C_3\xi_\uparrow(x_1)}dx_1=$$

$$\frac{C_{17}}{\varepsilon^{l+1}}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)+2C_3\xi_\uparrow(x)}\int\limits_{\xi_\uparrow(x)}^{+\infty}\xi^l e^{-2C_3\xi}d\xi<+\infty.$$

Let us now find the asymptotics of integrals, contained in (56) and (57). Due to (37) and (38) the integrand of $\int\limits_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1$ reads

$$\hat\varphi_{0,\uparrow}(x)\varphi_{l,\uparrow}(x)=\frac{1}{2^l l!\sqrt{V(x)-\lambda}}\Big(\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big)^l\Big[1+O\Big(\frac{\ln\eta_\uparrow(x)}{\eta_\uparrow(x)}\Big)\Big],\quad x\to+\infty. \tag{58}$$

The first term of right side of (58) contributes into the integral as follows,

$$\frac{1}{2^l l!}\int\limits_{R_1}^{x}\Big(\int\limits_{R_1}^{x_1}\frac{dx_2}{\sqrt{V(x_2)-\lambda}}\Big)^l\frac{dx_1}{\sqrt{V(x_1)-\lambda}}=\frac{2}{(l+1)!}\Big(\frac12\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big)^{l+1}, \tag{59}$$

and the absolute value of contribution of the second term is less than or equal to

$$C_{18}\int\limits_{R_1}^{x}\Big(\int\limits_{R_1}^{x_1}\frac{dx_2}{\sqrt{|V(x_2)-\lambda|}}\Big)^l\frac{\ln(2+\eta_\uparrow(x_1))}{2+\eta_\uparrow(x_1)}\frac{dx_1}{\sqrt{|V(x_1)-\lambda|}},$$

for a constant $C_{18}>0$. In view of Lemma 5 the latter expression is less than or equal to

$$C_{19}\int\limits_{R_0}^{x}\Big(\int\limits_{R_0}^{x_1}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l\frac{\ln(2+\eta_\uparrow(x_1))}{2+\eta_\uparrow(x_1)}\frac{dx_1}{\sqrt{|V(x_1)|}}\leqslant$$

$$C_{19}\int\limits_{R_0}^{x}\eta_\uparrow'(x_1)(2+\eta_\uparrow(x_1))^{l-1}\ln(2+\eta_\uparrow(x_1))\,dx_1\leqslant$$

$$C_{19}(2+\eta_\uparrow(x))^l\ln(2+\eta_\uparrow(x))=O\Big[\frac{\ln\eta_\uparrow(x)}{\eta_\uparrow(x)}\Big(\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{l+1}\Big]=$$

$$O\Big[\frac{\ln\eta_\uparrow(x)}{\eta_\uparrow(x)}\Big(\int\limits_{R_1}^{x}\frac{\mathrm{Re}\,\sqrt{V^*(x_1)-\lambda^*}}{|V(x_1)-\lambda|}\,dx_1\Big)^{l+1}\Big]=$$

$$O\Big[\frac{\ln\eta_\uparrow(x)}{\eta_\uparrow(x)}\Big(\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big)^{l+1}\Big],\qquad x\to+\infty, \tag{60}$$

for a constant $C_{19}>0$. Thus,

$$\int\limits_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1=\frac{2}{(l+1)!}\Big(\frac12\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big)^{l+1}\Big[1+O\Big(\frac{\ln\eta_\uparrow(x)}{\eta_\uparrow(x)}\Big)\Big],\quad x\to+\infty. \tag{61}$$

In the case $l=0$ one may write (58), using (17), (19) and Lemma 6 in the form

$$\hat{\varphi}_{0,\uparrow}(x)\varphi_{0,\uparrow}(x)=\frac{1}{\sqrt{V(x)-\lambda}}\Big[1+O\Big(\frac{1}{\xi_\uparrow(x)}\Big)\Big],\qquad x\to+\infty. \tag{62}$$

Respectively, due to Lemma 5 and for $V(x)\in\mathcal{K}$ the contribution of the second term of (62) to the integral $\int\limits_{R_1}^{x}\hat{\varphi}_{0,\uparrow}(x_1)\varphi_{0,\uparrow}(x_1)\,dx_1$ is less than or equal to,

$$C_{20}\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{|V(x_1)-\lambda|}(\xi_0+\xi_\uparrow(x_1))}\leqslant C_{20}\sqrt[4]{C_2}\int\limits_{R_0}^{x}\frac{dx_1}{\sqrt{|V(x_1)|}(\xi_0+\int_{R_0}^{x_1}\sqrt{|V(x_2)|}\,dx_2)}\leqslant$$

$$C_{20}\sqrt[4]{C_2}\int\limits_{R_0}^{x}\frac{\eta'_\uparrow(x_1)\,dx_1}{\xi_0+\varepsilon\eta_\uparrow(x_1)}=C_{20}\sqrt[4]{C_2}\frac{1}{\varepsilon}\ln[1+\varepsilon\eta_\uparrow(x)/\xi_0]=O(\ln\eta_\uparrow(x))=$$

$$O\Big(\frac{\ln\eta_\uparrow(x)}{\eta_\uparrow(x)}\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big),\qquad x\to+\infty,$$

for constants $C_{20}>0$ and $\xi_0>0$.

In view of (37), (40), Lemma 5 and $V(x)\in\mathcal{K}$, the following estimate holds for the integral $\int\limits_{+\infty}^{x}\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1$, [4]

$$\Big|\int\limits_{+\infty}^{x}\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1\Big|\leqslant C_{20}\int\limits_{x}^{+\infty}\Big(\int\limits_{R_1}^{x_1}\frac{dx_2}{\sqrt{|V(x_2)-\lambda|}}\Big)^{l}e^{-2\mathrm{Re}\,\xi_\uparrow(x_1;\lambda)}\frac{dx_1}{\sqrt{|V(x_1)-\lambda|}}\leqslant$$

$$C_{21}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)+C_{22}\xi_\uparrow(x)}\int\limits_{x}^{+\infty}\Big(\int\limits_{R_1}^{x_1}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^{l}\frac{e^{-C_{22}\xi_\uparrow(x_1)}\,dx_1}{\sqrt{|V(x_1)|}}=$$

$$-\frac{C_{21}}{C_{22}}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)+C_{22}\xi_\uparrow(x)}\Bigg\{\frac{e^{-C_{22}\xi_\uparrow(x_1)}}{|V(x_1)|}\eta^l_\uparrow(x_1)\Big|_x^{+\infty}-$$

$$\int\limits_{x}^{+\infty}e^{-C_{22}\xi_\uparrow(x_1)}\Big[\frac{l\eta_\uparrow^{l-1}(x_1)}{|V(x_1)|^{3/2}}-\frac{\mathrm{Re}\,V(x_1)\mathrm{Re}\,V'(x_1)+\mathrm{Im}\,V(x_1)\mathrm{Im}\,V'(x_1)}{|V(x_1)|^3}\eta^l_\uparrow(x_1)\Big]dx_1\Bigg\}\leqslant$$

$$\frac{C_{21}}{C_{22}}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big[\frac{\eta^l_\uparrow(x)}{|V(x)|}+e^{C_{22}\xi_\uparrow(x)}\int\limits_{x}^{+\infty}\frac{e^{-C_{22}\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\Big(\frac{l}{\varepsilon}\eta_\uparrow^{l-1}(x_1)+C_{23}\frac{\eta^l_\uparrow(x_1)}{\xi_\uparrow(x_1)}\Big)dx_1\Big]\leqslant$$

$$\frac{C_{21}}{\varepsilon C_{22}}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big[\eta^l_\uparrow(x)+e^{C_{22}\xi_\uparrow(x)}\int\limits_{x}^{+\infty}\frac{e^{-C_{22}\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\eta_\uparrow^{l-1}(x_1)(l+C_{23})\,dx_1\Big]=$$

$$O[e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\eta^l_\uparrow(x)]+O\Big[e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)+C_{22}\xi_\uparrow(x)}\frac{1}{\eta_\uparrow(x)}\int\limits_{x}^{+\infty}\frac{e^{-C_{22}\xi_\uparrow(x_1)}}{\sqrt{|V(x_1)|}}\eta^l_\uparrow(x_1)\,dx_1\Big]=$$

[4] In (63) and (65) the estimate $\int_{R_1}^{x}dx_1/\sqrt{|V(x_1)|}=O(\int_{R_1}^{x}dx_1/\sqrt{V(x_1)-\lambda})$ is used. Derivation of this estimate is contained in (60).

$$O\Big[e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big(\int\limits_{R_1}^{x}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l\Big]=O\Big[e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big(\int\limits_{R_1}^{x}\frac{dx_2}{\sqrt{V(x_2)-\lambda}}\Big)^l\Big],\qquad x\to+\infty,\tag{63}$$

for positive constants $C_{20}, \ldots, C_{23}$ The asymptotics (37) and (39) for $n=l+1$ is derived from (56), (61), (63) from (37) and (38) for $n=0$ and from (18), (20), (40) and corollary 1.

The integral $\int\limits_{R_1}^{x}\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1$ can be calculated in the same way as the following one, $\int\limits_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\varphi_{l,\uparrow}(x_1)\,dx_1$ and the result is

$$\int\limits_{R_1}^{x}\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1=-\frac{2}{(l+1)!}\Big(-\frac{1}{2}\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big)^{l+1}\Big[1+O\Big(\frac{\ln\eta_\uparrow(x)}{\eta_\uparrow(x)}\Big)\Big],\ \ x\to+\infty.\tag{64}$$

For the integral $\int\limits_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1$, due to (38), (40), Lemma 5 and $V(x)\in\mathcal{K}$, the estimate takes place,

$$\Big|\int\limits_{R_1}^{x}\hat\varphi_{0,\uparrow}(x_1)\hat\varphi_{l,\uparrow}(x_1)\,dx_1\Big|\leqslant C_{24}\int\limits_{R_1}^{x}\Big(\int\limits_{R_1}^{x_1}\frac{dx_2}{\sqrt{|V(x_2)-\lambda|}}\Big)^l\frac{e^{2\mathrm{Re}\,\xi_\uparrow(x_1;\lambda)}\,dx_1}{\sqrt{|V(x_1)-\lambda|}}\leqslant$$

$$C_{25}e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)-C_{26}\xi_\uparrow(x)}\int\limits_{R_0}^{x}\Big(\int\limits_{R_0}^{x_1}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l\frac{e^{C_{26}\xi_\uparrow(x_1)}\,dx_1}{\sqrt{|V(x_1)|}}=$$

$$\frac{C_{25}}{\varepsilon}e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)-C_{26}\xi_\uparrow(x)}\eta_\uparrow^l(x)\int\limits_{R_0}^{x}\xi_\uparrow'(x_1)e^{C_{26}\xi_\uparrow(x_1)}\,dx_1\leqslant\frac{C_{25}}{\varepsilon C_{26}}e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\eta_\uparrow^l(x)=$$

$$O\Big[e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big(\int\limits_{R_1}^{x}\frac{dx_2}{\sqrt{|V(x_2)|}}\Big)^l\Big]=O\Big[e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big(\int\limits_{R_1}^{x}\frac{dx_2}{\sqrt{V(x_2)-\lambda}}\Big)^l\Big],\qquad x\to+\infty,\tag{65}$$

for positive constants $C_{24},\ldots, C_{27}$. The asymptotics (38) for $n=l+1$ follows from (57), (64), (65) as well as from (37) and (38) for $n=0$.

Finally let us check that $\varphi_{n,\uparrow}(x)$ ($\hat\varphi_{n,\uparrow}(x)$) for any n is normalizable (non-normalizable) at $+\infty$. For this purpose it is sufficient to examine that the leading term of the right side (37) ((38)) is normalizable (non-normalizable) at $+\infty$. Normalizability of the leading term of (37) follows from the fact that due to $V(x)\in\mathcal{K}$ and Lemma 5 the following estimate is valid,

$$\frac{1}{\sqrt{|V(x)-\lambda|}}e^{-2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big|\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big|^{2n}\leqslant\frac{C_2^{(2n+1)/4}}{\sqrt{|V(x)|}}e^{-2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))}\times$$

$$\Big(\int\limits_{R_0}^{x}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{2n}\leqslant\frac{C_2^{(2n+1)/4}\xi_\uparrow'(x)}{\varepsilon^{2n}|V(x)|}e^{-2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))}\xi_\uparrow^{2n}(x)\leqslant$$

$$\frac{C_2^{(2n+1)/4}}{\varepsilon^{2n+1}}\xi_\uparrow^{2n}(x)\xi_\uparrow'(x)e^{-2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))},\qquad x\geqslant R_1,$$

the right side of which is obviously normalizable at $+\infty$. Non-normalizability of the leading term in (38) follows from the fact that in view of (25), Lemma 5 and with the help of the trick in (55) the following estimate holds:

$$\frac{1}{\sqrt{|V(x)-\lambda|}}e^{2\mathrm{Re}\,\xi_\uparrow(x;\lambda)}\Big|\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{V(x_1)-\lambda}}\Big|^{2n}\geqslant\frac{C_1^{(4n+1)/4}C_3^{2n}}{\sqrt{|V(x)|}}\frac{\xi'_\uparrow(x)}{\sqrt{|V(x)|}}e^{2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))}\times$$

$$\Big(\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{2n}\geqslant\frac{C_1^{(4n+1)/4}C_3^{2n}}{C_0}\frac{\xi'_\uparrow(x)}{(\xi_0+\xi_\uparrow(x))^\gamma}e^{2C_3(\xi_\uparrow(x)-\xi_\uparrow(R_1))}\Big(\int\limits_{R_1}^{x}\frac{dx_1}{\sqrt{|V(x_1)|}}\Big)^{2n},\ x\geqslant R_1,$$

the right side of which is evidently non-normalizable at $+\infty$. Lemma 9 is proved.

Corollary 4. In conditions of the Lemma 9 any formal associated function of h of n-th order normalizable at $\pm\infty$, for a spectral value λ such that either $\lambda\leqslant 0$ or $\mathrm{Im}\,\lambda\neq 0$, can be written in the form

$$\sum_{j=0}^{n}a_{j,\uparrow\downarrow}\varphi_{j,\uparrow\downarrow}(x),\qquad a_{j,\uparrow\downarrow}=\mathrm{Const},\quad a_{n,\uparrow\downarrow}\neq 0 \tag{66}$$

and any associated function of h of n-th order, non-normalizable at $\pm\infty$, for the same spectral value λ can be presented as follows

$$\sum_{j=0}^{n}\big(b_{j,\uparrow\downarrow}\varphi_{j,\uparrow\downarrow}(x)+c_{j,\uparrow\downarrow}\hat\varphi_{j,\uparrow\downarrow}(x)\big), \tag{67}$$

where $b_{j,\uparrow\downarrow},c_{j,\uparrow\downarrow}=\mathrm{Const}$ and either $b_{n,\uparrow\downarrow}\neq 0$ or $c_{n,\uparrow\downarrow}\neq 0$.

Corollary 5. For normalizable associated functions $\psi_1(x)$ and $\psi_2(x)$ of a Hamiltonian $h\in K$ of any orders, for eigenvalues λ_1 and λ_2 respectively such that either $\lambda_{1,2}\leqslant 0$ or $\mathrm{Im}\,\lambda_{1,2}\neq 0$ the equality

$$\int\limits_{-\infty}^{+\infty}[h\psi_1(x)]\psi_2(x)\,dx=\int\limits_{-\infty}^{+\infty}\psi_1(x)[h\psi_2(x)]\,dx \tag{68}$$

takes place.

5. Invariance of the potential sets $\mathcal{K}$ and K

Invariance of the potential sets $\mathcal{K}$ and K under intertwining is proved in Lemmas 1 and 10 respectively.

Lemma 10. *Let: 1) $h^+=-\partial^2+V_1(x)$, $V_1(x)\in\mathcal{K}$; 2) $\lambda\in\mathbb{C}$ and either $\lambda\leqslant 0$ or $\mathrm{Im}\,\lambda\neq 0$; 3) $\varphi(x)$ be zero-mode of $h^+-\lambda$; 4) $\chi(x)=-\varphi'(x)/\varphi(x)$, $q_1^\pm=\mp\partial+\chi(x)$. Then the potential $V_2(x)$ of the Hamiltonian*

$$h^-\equiv-\partial^2+V_2(x)=\lambda+q_1^-q_1^+,$$

intertwined with $h^+=\lambda+q_1^+q_1^-$ by means of equalities

$$q_1^\pm h^\mp=h^\pm q_1^\pm$$

belongs to $\mathcal{K}$ also.

Proof.

Let us first check that there is $R'_{02} > 0$ such that

$$V_2(x) \equiv V_1(x) - 2(\ln \varphi(x))''$$

(see Eq. (53) in [7]) for $|x| \geqslant R'_{02}$ is twice continuously differentiable. For this purpose it is sufficient to show that there is $R'_{02} > 0$ such that $\varphi(x)$ for $|x| \geqslant R'_{02}$ has not zeroes and is four times continuously differentiable. Existence of $R'_{02} > 0$ such that $\varphi(x)$ for $|x| \geqslant R'_{02}$ has not zeroes follows from the fact that one of asymptotics of Lemma 8 is valid for (normalized) $\varphi(x)$. Without loss of generality suppose that this R'_{02} is so large that

$$V_1(x)\Big|_{[R'_{02},+\infty[} \in C^2_{[R'_{02},+\infty[}, \qquad V_1(x)\Big|_{]-\infty,-R'_{02}]} \in C^2_{]-\infty,-R'_{02}]}. \tag{69}$$

Then the fact that $\varphi(x)$ is four times continuously differentiable for $|x| \geqslant R'_{02}$ follows from the equality $\varphi'' = (V_1 - \lambda)\varphi$, from (69) and from the fact that $\varphi(x)$ is twice continuously differentiable for $|x| \geqslant R'_{02}$ as a zero-mode of $h^+ - \lambda$.

Let us now verify that $\operatorname{Im} V_2/\operatorname{Re} V_2 = o(1)$, $x \to \pm\infty$ and there are $R_{02} \geqslant R'_{02}$ and $\varepsilon_2 > 0$ such that $\operatorname{Re} V_2(x) \geqslant \varepsilon_2$ for any $|x| \geqslant R_{02}$. The former follows from (31) in view of $\operatorname{Im} V_1/\operatorname{Re} V_1 = o(1)$, $x \to \pm\infty$. Moreover, since obviously

$$\operatorname{Re} V_2(x) = \operatorname{Re} V_1(x)[1 + o(1)], \qquad x \to \pm\infty \tag{70}$$

and there are $R_{02} \geqslant R'_{02}$ and $\varepsilon_1 > 0$ such that for any $|x| \geqslant R_{02}$, the value of $[1 + o(1)]$ in (70) is more than or equal to $1/2$ and $\operatorname{Re} V_1(x) \geqslant \varepsilon_1$, so that for any $|x| \geqslant R_{02}$ the inequalities hold

$$\operatorname{Re} V_2(x) \geqslant \frac{1}{2}\operatorname{Re} V_1(x) \geqslant \frac{\varepsilon_1}{2},$$

wherefrom the existence of the required R_{02} and $\varepsilon_2 = \varepsilon_1/2$ follows.

Finally we show that the function

$$\Big(\int\limits_{R_{02}}^{x} \sqrt{|V_2(x_1)|}\, dx_1\Big)^2 \Big(\frac{|V_2'(x)|^2}{|V_2(x)|^3} + \frac{|V_2''(x)|}{|V_2(x)|^2}\Big) \tag{71}$$

is bounded for $x \geqslant R_{02}$ (the case with a similar function for $x \leqslant -R_{02}$ can be considered analogously). In view of $V_1(x) \in \mathcal{K}$, (28), (29) and (31) we have

$$\int\limits_{R_{02}}^{x} \sqrt{|V_2(x_1)|}\, dx_1 = O\big(\xi_{1,\uparrow}(x)\big), \qquad \frac{|V_2'(x)|^2}{|V_2(x)|^3} = O\Big(\frac{1}{\xi^2_{1,\uparrow}(x)}\Big),$$

$$\frac{|V_2''(x)|}{|V_2(x)|^2} = O\Big(\frac{1}{\xi^2_{1,\uparrow}(x)}\Big), \qquad x \to +\infty, \qquad \xi_{1,\uparrow}(x) = \int\limits_{R_{02}}^{x} \sqrt{|V_1(x)|} dx_1,$$

wherefrom boundedness of (71) is derived. Lemma 10 is proved.

Corollary 6. Using (18), (20) and Remark 1, (34) and (36) ((37) and (39)), (25), (66) and estimations similar to the estimations in the proof of Lemma 9, one can easily check that under conditions of Lemma 10 the operator q_1^- maps any formal eigenfunction or associated function (of any order) of the Hamiltonian h^+ normalizable at $+\infty$ (at $-\infty$)

to a function normalizable at $+\infty$ (at $-\infty$), for any spectral value λ' such that either $\lambda' \leqslant 0$ or $\operatorname{Im}\lambda' \neq 0$.

Lemma 1. *Let: 1) $h^+ = -\partial^2 + V_1(x)$, $V_1(x) \in K$; 2) $h^- = -\partial^2 + V_2(x)$, $V_2(x) \in C_{\mathbb{R}}$; 3) $q_N^- h^+ = h^- q_N^-$, where q_N^- is a differential operator of Nth order with coefficients belonging to $C_{\mathbb{R}}^2$; 4) each eigenvalue of $\mathbf{S}^+$-matrix of q_N^- (see Th. 1 in Part I [7]) satisfies one of the conditions: either $\lambda \leqslant 0$ or $\operatorname{Im}\lambda \neq 0$. Then: 1) $V_2(x) \in K$; 2) coefficients of q_N^- belong to $C_{\mathbb{R}}^\infty$; 3) $h^+ q_N^+ = q_N^+ h^-$, where $q_N^+ = (q_N^-)^t$, and moreover coefficients of q_N^+ belong to $C_{\mathbb{R}}^\infty$ as well.*

Proof.

Let $\varphi_1(x), \ldots, \varphi_N(x)$ be a basis in $\ker q_N^-$, in which $\mathbf{S}^+$-matrix of q_N^- (see Theorem 1 in Part I [7]) has the canonical form. Since, firstly, $\varphi_1(x), \ldots, \varphi_N(x)$ as eigen- and associated functions of h^+ belong to $C_{\mathbb{R}}^\infty$, secondly, the Wronskian $W(x)$ of the functions $\varphi_1(x), \ldots, \varphi_N(x)$ has not any zeros and, thirdly,

$$q_N^- = \frac{1}{W(x)} \begin{vmatrix} \varphi_1(x) & \varphi_1'(x) & \ldots & \varphi_1^{(N)}(x) \\ \varphi_2(x) & \varphi_2'(x) & \ldots & \varphi_2^{(N)}(x) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_N(x) & \varphi_N'(x) & \ldots & \varphi_N^{(N)}(x) \\ 1 & \partial & \ldots & \partial^N \end{vmatrix},$$

the coefficients of q_N^- and thereby of q_N^+ belong to $C_{\mathbb{R}}^\infty$. Belonging of $V_2(x)$ to $C_{\mathbb{R}}^\infty$ follows from the equality $V_2(x) = V_1(x) - 2(\ln W(x))''$ (see Eq. (53) in [7]), from inclusion $W(x) \in C_{\mathbb{R}}^\infty$ and from absence of zeroes for $W(x)$. Inclusion $V_2(x) \in K$ follows from inclusion $V_2(x) \in C_{\mathbb{R}}^\infty$, from Lemma 10 and can be also justified by the factorization procedure described in Lemma 1 of [6]. The equality $h^+ q_N^+ = q_N^+ h^-$ is obvious. Lemma 1 is proved.

6. Proofs of Lemmas 2–4 and Theorem 3

The properties of associated functions under intertwining are described by the

Lemma 2. *Let: 1) the conditions of the Lemma 1 take place; 2) $\varphi_n(x)$, $n = 0, \ldots M$ be a sequence of formal associated functions of h^+ for spectral value λ:*

$$h^+\varphi_0 = \lambda\varphi_0, \qquad (h^+ - \lambda)\varphi_n = \varphi_{n-1}, \quad n \geqslant 1,$$

where either $\lambda \leqslant 0$ or $\operatorname{Im}\lambda \neq 0$. Then:

1) there is a number m such that $0 \leqslant m \leqslant \min\{M+1, N\}$,

$$q_N^- \varphi_n \equiv 0, \qquad n < m$$

and

$$\psi_l = q_N^- \varphi_{m+l}, \qquad l = 0, \ldots, M - m$$

is a sequence of formal associated functions of h^- for the spectral value λ:

$$h^-\psi_0 = \lambda\psi_0, \qquad (h^- - \lambda)\psi_l = \psi_{l-1}, \quad l \geqslant 1;$$

2) if a function $\varphi_n(x)$, for a given $0 \leqslant n \leqslant M$, is normalizable at $+\infty$ (at $-\infty$), then $q_N^- \varphi_n$ is normalizable at $+\infty$ (at $-\infty$) as well.

Proof.

Existence of m such that $0 \leqslant m \leqslant \min\{M+1, N\}$,

$$q_N^- \varphi_n \equiv 0, \qquad n < m$$

and either $m > M$ or

$$q_N^- \varphi_m \not\equiv 0, \tag{72}$$

can be derived from linear independence of φ_n and from the fact that dimension of $\ker q_N^-$ is N. The fact that $\psi_l = q_N^- \varphi_{m+l}$, $l = 0, \ldots, M - m$ is a sequence of formal eigenfunction and associated functions of h^- (if $m \leqslant M$):

$$h^- \psi_0 = \lambda \psi_0, \qquad (h^- - \lambda)\psi_l = \psi_{l-1}, \quad l \geqslant 1,$$

follows from the chains:

$$h^- \psi_0 = h^- q_N^- \varphi_m = q_N^- h^+ \varphi_m = q_N^- (\lambda \varphi_m + \varphi_{m-1}) = \lambda \psi_0, \qquad \varphi_{-1} \equiv 0,$$

$$(h^- - \lambda)\psi_l = (h^- - \lambda) q_N^- \varphi_{m+l} = q_N^- (h^+ - \lambda) \varphi_{m+l} = q_N^- \varphi_{m+l-1} = \psi_{l-1}, \qquad l \geqslant 1,$$

if intertwining $h^- q_N^- = q_N^- h^+$ and (72) are used. Before the proof of the second statement of the Lemma 2 let us note that with the help of similar arguments one can show that in the conditions of Lemma 10 the operator q_1^- maps any formal eigenfunction or associated function of h^+ for a spectral value λ' either to the identical zero or to a formal eigenfunction or associated function of h^- for the same spectral value λ'. Thus, the second statement of the Lemma 2 follows from Lemma 10, corollary 6 and the construction, described in Lemma 1 of [6]. Lemma 2 is proved.

Corollary 7 (2). Since h^+ is an intertwining operator to itself and both eigenvalues of its $\mathbf{S}^+$-matrix (see Theorem 1 in Part I [7]) are zero, then if $\varphi_n(x)$ is normalizable at $+\infty$ (at $-\infty$), then $\varphi_j(x)$, $j = 0, \ldots n-1$ is normalizable at $+\infty$ (at $-\infty$) as well.

Corollary 8 (3). If there is a normalizable associated function of n-th order $\varphi_n(x)$ of the Hamiltonian h with a potential belonging to K for an eigenvalue λ, which is either $\lambda \leqslant 0$ or $\operatorname{Im} \lambda \neq 0$, then for this eigenvalue there is an associated function $\varphi_j(x)$ of the Hamiltonian h, normalizable on the whole axis, of any smaller order j:

$$\varphi_j = (h - \lambda)^{n-j} \varphi_n, \qquad j = 0, \ldots, n-1.$$

Corollary 9 (4). Let $\varphi_{i,j}^-(x)$ be a canonical basis of zero-modes of the intertwining operator q_N^-, *i.e.* such that $\mathbf{S}^+$-matrix (in Theorem 1 of Part I [7]) has in this basis the canonical (Jordan) form:

$$h^+ \varphi_{i,0}^- = \lambda_i \varphi_{i,0}^-, \quad (h^+ - \lambda_i)\varphi_{i,j}^- = \varphi_{i,j-1}^-, \quad i = 1, \ldots, n, \quad j = 1, \ldots, k_i - 1, \qquad \sum_{i=1}^n k_i = N.$$

Then there are numbers $k_{i\uparrow}^+$ and $k_{i\downarrow}^+$, $0 \leqslant k_{i\uparrow,\downarrow}^+ \leqslant k_i$ such that for any i the functions

$$\varphi_{i,j}^-(x), \qquad j = 0, \ldots, k_{i\uparrow,\downarrow}^+ - 1$$

are normalizable at $+\infty$ or $-\infty$ respectively and the functions

$$\varphi_{i,j}^-(x), \qquad j = k_{i\uparrow,\downarrow}^+, \ldots, k_i - 1$$

are non-normalizable at the same $+\infty$ or $-\infty$.

Independence of these numbers $k^{+}_{i\uparrow,\downarrow}$ on a choice of the canonical basis in the case, when the intertwining operator q_N^- cannot be stripped-off, follows from

Lemma 3. *Let: 1) conditions of Lemma 1 take place; 2) q_N^- not be able to be stripped-off. Then any two formal associated functions of h^+ of the same order for the same spectral value λ when being zero-modes of q_N^- are either simultaneously normalizable at $+\infty$ or simultaneously non-normalizable at $+\infty$. The same takes place at $-\infty$.*

Proof.

Assume that there are two sequences of a formal eigenfunction and associated functions of h^+ for the same spectral value λ:

$$h^+\phi_{l,0} = \lambda\phi_{l,0}, \qquad (h^+ - \lambda)\phi_{l,j} = \phi_{l,j-1}, \quad l = 1,2, \quad j = 1,\dots,j_0,$$

such that ϕ_{1,j_0} is normalizable at $+\infty$, ϕ_{2,j_0} is non-normalizable at $+\infty$ and

$$q_N^-\phi_{l,j_0} = 0, \qquad l = 1,2.$$

Let us show that it leads to contradiction.

Let us check first that

$$q_N^-\phi_{l,j} = 0, \qquad l = 1,2, \quad j = 0,\dots,j_0 - 1.$$

For $j = j_0 - 1$ these equalities follow from the chain

$$q_N^-\phi_{l,j_0-1} = q_N^-(h^+ - \lambda)\phi_{l,j_0} = (h^- - \lambda)q_N^-\phi_{l,j_0} = 0,$$

and for $j < j_0 - 1$ they can be derived in the same way by induction. As for intertwining operator, which cannot be stripped-off, there is only one zero-mode of $h^+ - \lambda$ (up to a constant cofactor), corresponding to a fixed eigenvalue λ of its $\mathbf{S}^+$-matrix (see Th. 1&2 in Part I [7]), so $\phi_{1,0}(x)$ and $\phi_{2,0}(x)$ are proportional. Without loss of generality suppose that $\phi_{1,j}$ and $\phi_{2,j}$ are normalized so that

$$\phi_{1,0}(x) \equiv \phi_{2,0}(x).$$

Then the sequence $\phi_{1,j} - \phi_{2,j}$ represents a sequence of associated functions of h^+ for the same eigenvalue λ (being zero-modes of q_N^-) and $\phi_{1,j_0} - \phi_{2,j_0}$ is an associated function of the order $j_1 < j_0$ non-normalizable at $+\infty$. But on the other hand there is an associated function ϕ_{1,j_1} of h^+ normalizable at $+\infty$ (see corollary 7 (2)) of the order j_1 for an eigenvalue λ which is a zero-mode of q_N^-. Performing in the same way by induction, we come to the conclusion that intersection $\ker q_N^- \cap \ker(h^+ - \lambda)$ (dimension of which is 1 in view of Th. 2 of Part I [7]) contains non-trivial functions normalizable and non-normalizable at $+\infty$, the latter being impossible. The consideration of the $-\infty$ case is analogous. The Lemma 3 is proved.

The following Lemma 4 clarifies interrelation between the behavior at $\pm\infty$ of elements of canonical bases of mutually transposed intertwining operators.

Lemma 4. *Let: 1) conditions of Lemma 1 take place; 2) $\{\varphi^-_{i,j}\}$ and $\{\varphi^+_{i,j}\}$ are canonical bases of $\ker q_N^-$ and $\ker q_N^+$ respectively; 3) q_N^- cannot be stripped-off; 4) k_i is algebraic multiplicity of eigenvalue λ_i of $\mathbf{S}^+$-matrix (see Th. 1 of Part I [7]). Then for any i and j the function $\varphi_{i,j}(x)$ is normalizable (non-normalizable) at $+\infty$ if and only if $\psi_{i,k_i-j-1}(x)$ is non-normalizable (normalizable) at $+\infty$. The same takes place at $-\infty$.*

Proof.

In accordance with corollary 9 (4) for any i the basis $\varphi^-_{i,j}$ has the following structure:

$$\varphi^-_{i,j}, \qquad j = 0, \ldots, k^+_{i\uparrow,\downarrow} - 1$$

are normalizable at $\pm\infty$ and

$$\varphi^-_{i,j}, \qquad j = k^+_{i\uparrow,\downarrow}, \ldots, k_i - 1$$

are non-normalizable at $\pm\infty$, where $0 \leqslant k^+_{i\uparrow,\downarrow} \leqslant k_i$. Moreover in view of Lemma 3 the numbers $k^+_{i\uparrow,\downarrow}$ are independent of a choice of a canonical basis. To prove Lemma 4 it is sufficient to establish that for any i

$$\varphi^+_{i,j}, \qquad j = 0, \ldots, k_i - k^+_{i\uparrow,\downarrow} - 1 \tag{73}$$

are normalizable at $\pm\infty$ and

$$\varphi^+_{i,j}, \qquad j = k_i - k^+_{i\uparrow,\downarrow}, \ldots, k_i - 1 \tag{74}$$

are non-normalizable at $\pm\infty$.

Let $\varphi_{i,j,\uparrow\downarrow}$ be a sequence of a formal eigenfunction and associated functions of h^+ normalizable at $\pm\infty$, for a spectral value λ_i. Then because of Lemma 3

$$q^-_N \varphi_{i,j,\uparrow\downarrow} \not\equiv 0, \qquad j = k^+_{i\uparrow,\downarrow}, \ldots, k_i - 1.$$

On the other hand, by virtue of Lemma 2, the functions $q^-_N \varphi_{i,j,\uparrow\downarrow}$ form a sequence of a formal eigenfunction and associated functions of h^- for the same spectral value λ_i and for $j \leqslant k_i - 1$ represent zero-modes of q^+_N (since in virtue of Th. 1 of Part I [7], $q^+_N q^-_N$ is a polynomial of h^+, containing cofactor $(h^+ - \lambda_i)^{k_i}$). Moreover in view of Lemma 2 these functions are normalizable at $\pm\infty$. Thus, by virtue of Lemma 3 the functions (73) are normalizable at $\pm\infty$.

We prove now that the functions (74) are non-normalizable at $\pm\infty$. For this purpose, because of corollary 9 (4), it is sufficient to prove that $\varphi^+_{i,k_i - k^+_{i\uparrow,\downarrow}}$ is non-normalizable at $\pm\infty$. Let us consider factorization of q^-_N in the product of intertwining operators of first order in accordance with Lemma 1 of [6]:

$$q^-_N = r^-_N \, \ldots \, r^-_1,$$

where $r^-_1, \ldots, r^-_{k_i}$ are chosen so that

$$r^-_{j+1} \, \ldots \, r^-_1 \varphi^-_{i,j} = 0, \qquad j = 0, \ldots, k_i - 1.$$

Then for q^+_N there is a factorization

$$q^+_N = (r^-_1)^t \, \ldots \, (r^-_N)^t,$$

where the zero-mode of $(r^-_j)^t$ is evidently

$$(r^-_{j+1})^t \, \ldots \, (r^-_N)^t \varphi^+_{i,k_i - j}, \qquad j = 1, \ldots, k_i.$$

Suppose that $\varphi^+_{i,k_i - k^+_{i\uparrow,\downarrow}}$ is normalizable at $\pm\infty$. Then

$$(r^-_{k^+_{i,\uparrow\downarrow}+1})^t \, \ldots \, (r^-_N)^t \varphi^+_{i,k_i - k^+_{i\uparrow,\downarrow}}$$

($i.e.$ in view of corollary 6 the zero-mode of $(r^-_{k^+_{i\uparrow,\downarrow}})^t$) is normalizable at $\pm\infty$ as well. But this statement contradicts to the fact that

$$r^-_{k^+_{i\uparrow,\downarrow}-1}\ldots r^-_1\varphi^-_{i,k^+_{i\uparrow,\downarrow}-1}$$

($i.e.$ the zero-mode of $r^-_{k^+_{i\uparrow,\downarrow}}$) is normalizable (because of the same corollary 6) at $\pm\infty$. Thus, $\varphi^+_{i,k_i-k^+_{i\uparrow,\downarrow}}$ is non-normalizable at $\pm\infty$ and Lemma 4 is proved.

A more precise result on interrelation between Jordan structures of intertwined Hamiltonians and the behavior of transformation functions is contained in

Theorem 3. *Let: 1) the conditions of Lemma 4 take place; 2) $\nu_\pm(\lambda)$ is an algebraic multiplicity of an eigenvalue λ of $h^\pm$, i.e. the number of independent eigenfunctions and associated functions of $h^\pm$ normalizable on the whole axis; 3) if λ is not an eigenvalue of $\mathbf{S}^+$ (see Th. 1 of Part I [7]), then $n_+(\lambda)=n_-(\lambda)=n_0(\lambda)=0$ and if $\lambda=\lambda_i$, where λ_i is an eigenvalue of $\mathbf{S}^+$, then $n_\pm(\lambda_i)$ is a number of functions among $\varphi^\mp_{i,j}(x)$, $j=0,\ldots,$ k_i-1 normalizable at both infinities and $n_0(\lambda_i)$ is a number of functions among $\varphi^-_{i,j}(x)$ (or $\varphi^+_{i,j}(x)$), $j=0,\ldots,k_i-1$ normalizable only at one of infinities. Then for any λ such that either $\lambda\leqslant 0$ or $\operatorname{Im}\lambda\neq 0$ the equality*

$$\nu_+(\lambda)-n_+(\lambda)=\nu_-(\lambda)-n_-(\lambda)$$

takes place. Moreover if $n_0(\lambda)>0$ for some λ, then for this λ

$$\nu_+(\lambda)-n_+(\lambda)=\nu_-(\lambda)-n_-(\lambda)=0.$$

Proof. Let us first notice that if for the level of the Hamiltonian h^+ λ such that either $\lambda\leqslant 0$ or $\operatorname{Im}\lambda\neq 0$ there is an associated function of the l-th order normalizable on the whole axis, then for the same level λ, any associated function of h^+ of the l-th order normalizable at one of infinities is normalizable on the whole axis. This fact is easily verifiable in a way similar to the proof of Lemma 3. Thus, in the case $n_0(\lambda)>0$ there is no any associated function of h^+ normalizable on the whole axis, of the order $n_+(\lambda)$ (and consequently of any greater order) for the level λ. Hence in this case $\nu_+(\lambda)=n_+(\lambda)$. Moreover in view of Lemma 4 and of the symmetry between h^+ and h^- the equality $\nu_-(\lambda)=n_-(\lambda)$ holds for the case $n_0(\lambda)>0$ as well. Thus, for this case Theorem 3 is proved.

In the case, when λ does not belong to the spectrum of $\mathbf{S}^+$-matrix (see Th. 1 in Part I [7]) Theorem 3 follows from Lemma 2 and from the fact that an associated function of $h^\pm$, which corresponds to λ under consideration, cannot be zero-mode of $q_N^\mp$ (since in the opposite case this function would be linear combination of formal eigenfunctions and associated functions of $h^\pm$, whose eigenvalues belong to the spectrum of $\mathbf{S}^+$).

Consider now the case, when $\lambda=\lambda_i$ belongs to the spectrum of $\mathbf{S}^+$ and $n_0(\lambda_i)=0$. We shall prove the inequality

$$\nu_-(\lambda_i)-\nu_+(\lambda_i)+n_+(\lambda_i)\leqslant n_-(\lambda_i) \tag{75}$$

only, because the opposite inequality

$$\nu_+(\lambda_i)-\nu_-(\lambda_i)+n_-(\lambda_i)\leqslant n_+(\lambda_i)$$

follows from (75), Lemma 4 and the symmetry between h^+ and h^- (the statement of the theorem is derived from these inequalities). Since in the subcase $\nu_-(\lambda_i)-\nu_+(\lambda_i)+n_+(\lambda_i) \leqslant 0$ the inequality (75) is trivial, we shall consider below the subcase

$$\nu_-(\lambda_i) - \nu_+(\lambda_i) + n_+(\lambda_i) > 0 \tag{76}$$

only.

Let us show that there is a sequence $\hat{\varphi}_{i,j}(x)$ such that,

$$h^+\hat{\varphi}_{i,0} = \lambda_i\hat{\varphi}_{i,0}, \qquad (h^+ - \lambda_i)\hat{\varphi}_{i,j} = \hat{\varphi}_{i,j-1}, \quad j = 1, \ldots, \nu_-(\lambda_i) + n_+(\lambda_i) - 1$$

$$q_N^-\hat{\varphi}_{i,j} = 0, \qquad j = 0, \ldots, \nu_-(\lambda_i) + n_+(\lambda_i) - 1$$

and the functions $\hat{\varphi}_{i,j}$, $j = \nu_+(\lambda_i)$, $\ldots$, $\nu_-(\lambda_i) + n_+(\lambda_i) - 1$ are non-normalizable at both infinities. This sequence cannot contain more than k_i terms, since in the opposite case associated functions of this sequence of orders greater k_i-1 would be linear combinations of $\varphi^-_{i,j}$, the latter being impossible. Therefore, in view of Lemma 3, the number of associated functions of the sequence non-normalizable at both infinities cannot be greater than the number of functions non-normalizable at both infinities among $\varphi^-_{i,j}$ with fixed i,

$$\nu_-(\lambda_i) - \nu_+(\lambda_i) + n_+(\lambda_i) \leqslant n_-(\lambda_i),$$

that is required to be proved.

Consider a sequence of $\varphi_{i,j,\uparrow\downarrow}$ formal eigenfunction and associated functions of h^+ normalizable at $\pm\infty$, for the level λ_i (this sequence exists due of Lemma 9). First $\nu_+(\lambda_i)$ functions of this sequence are normalizable at both infinities (following the arguments used at the beginning of this proof). By virtue of (66) any of the functions $\varphi^-_{i,j}$, $j = 0$, $\ldots$, $n_+(\lambda_i)-1$ can be presented as a linear combination of $\varphi_{i,0,\uparrow\downarrow}$, $\ldots$, $\varphi_{i,n_+(\lambda_i)-1,\uparrow\downarrow}$. Moreover, due to linear independence of $\varphi^-_{i,0}$, $\ldots$, $\varphi^-_{i,n_+(\lambda_i)-1}$ the reverse is valid as well. Hence,

$$q_N^-\varphi_{i,j,\uparrow\downarrow} = 0, \qquad j = 0, \ldots, n_+(\lambda_i) - 1.$$

Moreover, in view of Lemmas 2 and 3 the functions

$$q_N^-\varphi_{i,j,\uparrow\downarrow}, \qquad j = n_+(\lambda_i), \ldots$$

are different from zero and form a sequence of a formal eigenfunction and associated functions of h^- normalizable at $\pm\infty$ for the level λ_i. Applying the arguments of the beginning of this proof one can show that the first $\nu_-(\lambda_i)$ terms of this sequence are normalizable at both infinities.

Using the sequence of formal associated functions $\varphi_{i,j,\uparrow\downarrow}$ one can construct another sequence of formal associated functions of h^+ for the same level λ_i,

$$\tilde{\varphi}_{i,j,\uparrow\downarrow} = \sum_{k=0}^{j} A_{i,k,\uparrow\downarrow}\varphi_{i,j-k,\uparrow\downarrow}, \qquad A_{i,k,\uparrow\downarrow} = \mathrm{Const}, \quad A_{i,0,\uparrow\downarrow} \neq 0.$$

This sequence as well as the sequence $\varphi_{i,j,\uparrow\downarrow}$ has the following properties:

- $\tilde{\varphi}_{i,j,\uparrow\downarrow}$, $j = 0$, $\ldots$ are normalizable at $\pm\infty$;
- $\tilde{\varphi}_{i,j,\uparrow\downarrow}$, $j = 0$, $\ldots$, $\nu_+(\lambda_i) - 1$ are normalizable at both infinities;
- $$q_N^-\tilde{\varphi}_{i,j,\uparrow\downarrow} = 0, \qquad j = 0, \ldots, n_+(\lambda_i) - 1; \tag{77}$$

- $q_N^-\tilde{\varphi}_{i,j,\uparrow\downarrow}$, $\quad j = n_+(\lambda_i), \ldots$ are different from zero and form the sequence of formal eigenfunction and associated functions of h^- normalizable at $\pm\infty$ for the level λ_i;
- $q_N^-\tilde{\varphi}_{i,j,\uparrow\downarrow}$, $\quad j = n_+(\lambda_i), \ldots, \nu_-(\lambda_i)+n_+(\lambda_i)-1$ are normalizable at both infinities.

One can choose constants $A_{i,k,\uparrow\downarrow}$ so that the required sequence $\hat{\varphi}_{i,j}$ can be written in the form

$$\hat{\varphi}_{i,j} = \tilde{\varphi}_{i,j,\uparrow} - \tilde{\varphi}_{i,j,\downarrow}.$$

Indeed, notice that from inequalities (76) and $\nu_+(\lambda_i) \geqslant n_+(\lambda_i)$ it follows that $\nu_-(\lambda_i) > 0$, *i.e.* that there is a normalizable eigenfunction $\psi_{i,0}$ of h^- for the level λ_i. As there is only one (up to constant cofactor) normalizable eigenfunction of h^- for the level λ_i the equalities

$$q_N^-\varphi_{i,n_+(\lambda_i),\uparrow\downarrow} = C_{i,\uparrow\downarrow}\psi_{i,0}$$

take place for some constants $C_{i,\uparrow\downarrow} \neq 0$. The fact that the relation

$$q_N^-\hat{\varphi}_{i,j} = 0 \tag{78}$$

holds for $j = 0, \ldots, n_+(\lambda_i)-1$ follows from (77). The equality (78) holds for $j = n_+(\lambda_i)$ if we take $A_{i,0,\uparrow\downarrow} = C_{i,\downarrow\uparrow}$ since

$$q_N^-\hat{\varphi}_{i,n_+(\lambda_i)} = \sum_{k=0}^{n_+(\lambda_i)} \Big(A_{i,k,\uparrow}q_N^-\varphi_{i,n_+(\lambda_i)-k,\uparrow} - A_{i,k,\downarrow}q_N^-\varphi_{i,n_+(\lambda_i)-k,\downarrow}\Big) =$$

$$A_{i,0,\uparrow}q_N^-\varphi_{i,n_+(\lambda_i),\uparrow} - A_{i,0,\downarrow}q_N^-\varphi_{i,n_+(\lambda_i),\downarrow} = \big(A_{i,0,\uparrow}C_{i,\uparrow} - A_{i,0,\downarrow}C_{i,\downarrow}\big)\psi_{i,0} = 0.$$

At last, one can attain validity of (78) for $j = n_+(\lambda_i)+1, \ldots, \nu_-(\lambda_i)+n_+(\lambda_i)-1$, looking through all $j = n_+(\lambda_i)+1, \ldots, \nu_-(\lambda_i)+n_+(\lambda_i)-1$ and taking into account at every step that $q_N^-\hat{\varphi}_{i,j}$ being normalizable eigenfunction of h^- is proportional to $\psi_{i,0}$. One has also to take into account that the dependence $q_N^-\hat{\varphi}_{i,j}$ of $A_{i,j-n_+(\lambda_i),\uparrow\downarrow}$ is linear,

$$q_N^-\hat{\varphi}_{i,j} = \sum_{k=0}^{j} \Big(A_{i,k,\uparrow}q_N^-\varphi_{i,j-k,\uparrow} - A_{i,k,\downarrow}q_N^-\varphi_{i,j-k,\downarrow}\Big)$$

$$= \sum_{k=0}^{j-n_+(\lambda_i)} \Big(A_{i,k,\uparrow}q_N^-\varphi_{i,j-k,\uparrow} - A_{i,k,\downarrow}q_N^-\varphi_{i,j-k,\downarrow}\Big)$$

$$= \sum_{k=0}^{j-n_+(\lambda_i)-1} \Big(A_{i,k,\uparrow}q_N^-\varphi_{i,j-k,\uparrow} - A_{i,k,\downarrow}q_N^-\varphi_{i,j-k,\downarrow}\Big) + \big(A_{i,j-n_+(\lambda_i),\uparrow}C_{i,\uparrow} - A_{i,j-n_+(\lambda_i),\downarrow}C_{i,\downarrow}\big)\psi_{i,0},$$

and choose $A_{i,j-n_+(\lambda_i),\uparrow\downarrow}$ so that the proportionality coefficient between $q_N^-\hat{\varphi}_{i,j}$ and $\psi_{i,0}$ is vanishing. It happens that among $\hat{\varphi}_{i,j}$ there are $\nu_-(\lambda_i)+n_+(\lambda_i)-\nu_+(\lambda_i)$ functions, which are non-normalizable at both infinities, because $\tilde{\varphi}_{i,j,\uparrow\downarrow}$, $j = \nu_+(\lambda_i), \ldots, \nu_-(\lambda_i)+n_+(\lambda_i)-1$ are normalizable at $\pm\infty$ only. Thus, the required sequence is constructed and Theorem 3 is proved.

7. Acknowledgements

I am grateful to my collaborators A.A.Andrianov and F.Cannata for invaluable remarks, fruitful discussions and reading the manuscript. This work was supported by Grant RFBR 06-01-00186-a and partially supported by the INFN grant.

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